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LOCAL MINIMA-BASED RECURRENCE PLOTS FOR CONTINUOUS DYNAMICAL SYSTEMS

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A major issue in using recurrence plots (RPs) to study dynamical systems is the choice of neighborhood size for thresholding the distance matrix that creates the plot. This is particularly important for continuous dynamical systems as temporal correlations of the trajectory might provide redundant information for recurrence analysis. We suggest an alternative procedure for creating RPs using the local minima provided by the distance profile, which approximately corresponds to the recurrence information in the orthogonal direction. The local minima-based thresholding yields a clean RP of minimized line thickness, that is compared to the plot obtained by the standard radius bases thresholding. New definitions of line segments arising from the local minima-based method are outlined, which yield consistent results with those derived from standard methods. Our preliminary comparison suggests that the newly introduced thresholding technique is more sensitive to small changes in a system's dynamics. We demonstrate our method via the chaotic Lorenz system without the loss of generality.

Keywords: Distance function; local minima thresholding; recurrence quantification analysis; continuous dynamical system.

1. Introduction

Recurrence is a fundamental property of most dynamical systems: as a system evolves in time, it comes arbitrarily close to points previously visited in phase space [Poincaré, 1890]. The pattern of recurring states is a meaningful source of information about the dynamics of the system [Marwan *et al.*, 2007; Webber Jr. & Zbilut, 2005]. Recurrence plots (RPs) and recurrence quantification analysis (RQA) are methods for visualizing and measuring the recurrent patterns exhibited by a dynamical system. The creation of a recurrence plot requires first calculating a distance matrix, followed by a thresholding procedure to qualify point pairs in phase space as being recurrent with one another. This paper focuses on a new method for performing this thresholding operation that takes into account the information provided by the local minima of the distance profile. Before going further we begin with a general overview of RPs and RQA.

A recurrence plot can be constructed from any time series, either by starting with an explicitly defined phase space or by reconstructing the phase space from an empirical signal [Takens, 1981; Abarbanel *et al.*, 1993]. The phase space reconstruction involves creating additional phase space dimensions by using time-delayed copies of the one-dimensional time series. Two parameters are used for this operation, the time delay τ and the embedding dimension m (necessary to completely "unfold" the attractor). The optimum embedding parameters for m and τ are often guided by approaches such as the false nearest neighbor and mutual information methods, respectively [Abarbanel *et al.*, 1993; Hegger *et al.*, 1999].

The first step of creating a recurrence plot is to construct the distance matrix by calculating the distance between every possible point pair in phase space (i.e. the state vectors). The result of this operation is an $N \times N$ matrix of distances where N is the number of points in the phase space. Given a trajectory of a dynamical system consisting of different states $\mathbf{x}_i \in \mathbb{R}^m$ (i.e. a sampled trajectory from system X of dimension m, where i indicates the time of observation), the corresponding distance matrix $\mathbf{D}$ is defined as follows

$$D_{i,j} = \|\mathbf{x}_i - \mathbf{x}_j\|, \quad i, j = 1, \ldots, N, \tag{1}$$

where $\|\cdot\|$ is the distance between two observations in phase space (e.g. Euclidean or Maximum norm). The dynamical behavior of an m-dimensional system is now mapped into a two-dimensional matrix. For illustration, we will use the Lorenz system henceforth [Lorenz, 1963]

$$\frac{d}{dt}(x, y, z) = (\sigma(y - x), x(\rho - z) - y, xy - \beta z), \tag{2}$$

with $\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. The distance matrix can be displayed as an image, where each axis shows the sampled time points of the phase space, and the distance between any two points is represented with a color map [Fig. 1(c)].

The distance matrix is symmetrical about the main diagonal (the *line of identity LOI*) by default since $D_{i,j} = D_{j,i}$. The distance topology of these various types of dynamical systems can be understood as ridges and valleys running at 45° angles through the distance plots. Valleys reveal similarities or recurrences of trajectories in the temporal evolution of the system. The majority of recurrence quantification measures as well as the estimation of dynamical invariants center around quantifying these diagonal line segments [Fig. 1(d)], but require a clear definition of what is recurrent and what is not. Therefore, each point in the distance matrix will be qualified as being either recurrent or nonrecurrent by thresholding the distance matrix. Points that are close to one another are considered as recurrent, and points that are far from one another as nonrecurrent, resulting in a binary matrix [Fig. 1(d)].

An ideal thresholding procedure is optimal insofar as it renders patterns of recurrence with utmost clarity. Keeping too many recurrence points results in a redundancy of recurrent information, and renders recurrence patterns less distinct. Keeping too few recurrence points results in a loss of recurrence information by removing recurrences that are informative of the system's dynamics. Moreover, the ideal recurrence criterion should not consider subsequent points on the same trajectory as recurrent, i.e. $\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots$, (which cause thickening of diagonal lines in the RP). We will refer to these subsequent points as *tangential motion*. In the case of discrete systems, thickness of diagonals are often related to laminar states which are helpful for the understanding of the system. However, in the case of continuous systems, they often provide redundant information related to temporal correlations of the trajectory. Hence, we should exclude the effects of the tangential motion on recurrence analysis at least for continuous systems. In this paper, we introduce an alternative way to threshold the distance matrix Eq. (1) by taking into account the local minima of the properly defined distance function. Therefore, in the following we focus on thresholding the distance matrix for continuous dynamical systems.

In Sec. 2, we will discuss some details about the standard thresholding techniques utilized in the literature. In Sec. 3, we present a local minima-based thresholding algorithm, which additionally (Sec. 4) requires a new definition for line segments. In Sec. 5, we show the efficiency of our method by distinguishing different dynamical regimes in the chaotic Lorenz system. Finally in Sec. 6, we draw some conclusions based on this work and discuss future directions of research.

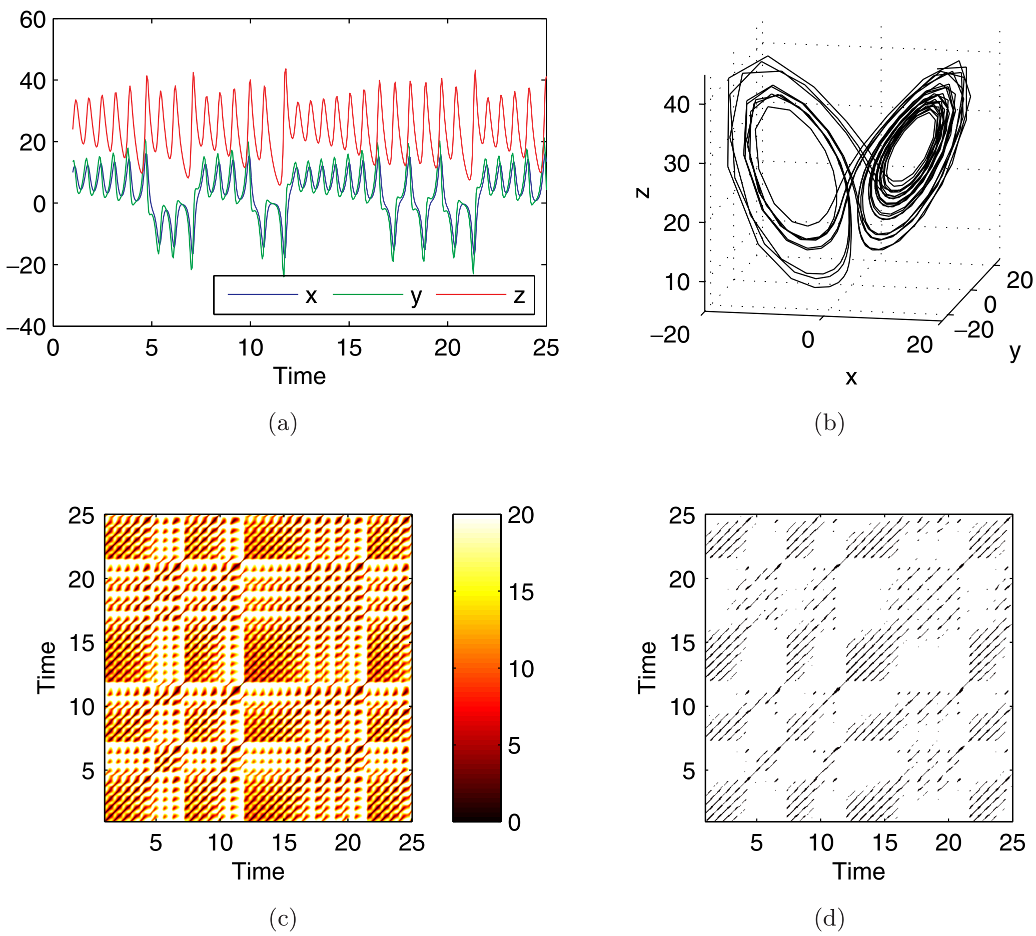


Fig. 1. (a) A realization of the Lorenz oscillator, (b) its phase space representation, (c) distance matrix and (d) recurrence plot (fixed threshold $\epsilon = 4$, Euclidean norm).

2. The Standard Thresholding Method for Creating Recurrence Plots

Recurrence quantification analysis, the estimation of dynamical invariants, or synchronization analysis require a binary RP [Marwan *et al.*, 2007]. The most commonly used thresholding procedure to obtain RPs involves the application of a distance threshold ϵ to the distance matrix [Figs. 1(c) and 1(d)]. Any pair of states $(\mathbf{x}_i, \mathbf{x}_j)$ in phase space whose distance is less than an ϵ value is considered to be recurrent, while any point pair whose distance is larger than this ϵ value is classified as nonrecurrent. This is formally expressed by

$$\mathbf{R}(\epsilon) = \Theta(\epsilon - \mathbf{D}), \tag{3}$$

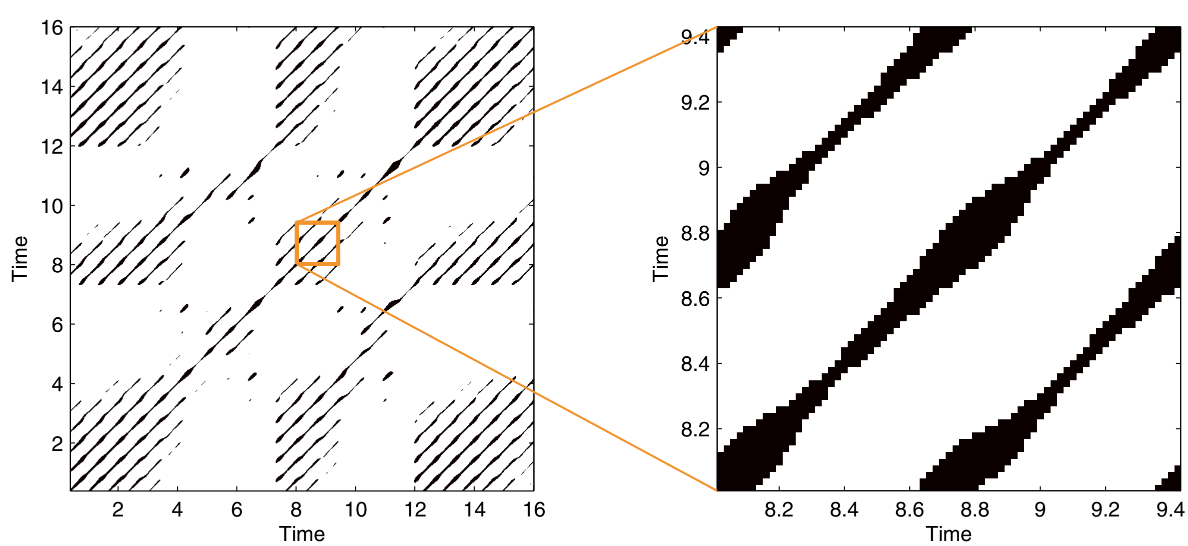
where Θ is the Heaviside function, $\mathbf{D}$ is the distance matrix and $\mathbf{R}$ is the recurrence matrix. The threshold ϵ spans a neighborhood around each state $\mathbf{x}_i$, and other states $\mathbf{x}_j$ falling into this neighborhood are considered to be recurrent. For our current purpose, we will focus only on the L_2 Euclidean distance metric with a fixed threshold ϵ, since this is the most widely used method. By visualizing this matrix with, black $(R_{i,j} = 1)$ and white $(R_{i,j} = 0)$ dots we see that different types of dynamics cause different types of line structures [Marwan *et al.*, 2007].

The goal of choosing ϵ is to retain as much unique and dynamically informative information as possible while minimizing the presence of redundant information that will negatively affect the appearance of the recurrence structure. These two goals are frequently difficult to satisfy simultaneously, which means choosing an appropriate ϵ can be difficult [Marwan, 2011]. Because the quantification of the recurrence structure is sensitive to the choice of ϵ, the parameter must be chosen with care. Several approaches have been suggested

[Marwan *et al.*, 2007; Schinkel *et al.*, 2008; Marwan, 2011]. Webber and Zbilut [2005] recommended using several ϵ values and then using the following three rules of thumb to choose an appropriate ϵ (for the definition of the following mentioned RQA measures we refer to [Marwan *et al.*, 2007]). (1) The ϵ value should lie in the scaling region visible by plotting ϵ values by RR in log–log coordinates. (2) The total number of recurrences should be kept low, in the range of 1–2%, and (3) The ϵ value may sometimes coincide with the first notch in the ϵ by DET graph, which can be used to further guide the choice of the ϵ. Determination of the optimal threshold follows from inspection and judgment of the above criteria. ϵ by DET graph indicates whether the inclusion of additional recurrent points via a larger ϵ value adds to the pre-existing deterministic structure as indicated by an increase in DET. The appropriate ϵ value lies in the first decrease of DET observed by increasing ϵ. However, this notch is not always present, and so cannot reliably be used to determine the ϵ value in every case. Other approaches suggest using ϵ values to cover 10% or even 5% of the maximal phase space diameter [Zbilut & Webber, 1992; Schinkel *et al.*, 2008] or to be 5–10% of the standard deviation [Marwan *et al.*, 2007].

The original definition of a RP uses a different criterion, where the number of points falling into the neighborhood of the state $\mathbf{x}_i$ is constant [Eckmann *et al.*, 1987]. Other recurrence criteria involve dynamical properties. For example, in perpendicular RPs two points $\mathbf{x}_i$ and $\mathbf{x}_j$ are recurrent if the jth point is within the neighborhood of i and lies on a plane that is perpendicular to the tangential flow at the ith vector point [Choi *et al.*, 1999]. In an iso-directional RP recurrences are related to trajectories which run parallel and in the same direction [Horai & Aihara, 2002]. Such approaches are refinements of the standard method that uses a definitional criteria to further restrict the set of points initially qualified as recurrent using a distance threshold in order to exclude recurrences coming from tangential motion. A review and comparison of different recurrence criteria is beyond the scope of this paper (see [Marwan *et al.*, 2007] for a more detailed discussion). Nevertheless, most of these methods need at least one parameter: the recurrence threshold ϵ or the number of nearest neighbors.

By taking a closer look at RPs, the issues surrounding the choice of an appropriate ϵ value become apparent. As ϵ is increased, more recurrent information is captured, but the thickness of the diagonal lines in the RP also increases. The thickness of diagonals is sometimes related to laminar states of the system [Marwan *et al.*, 2002; Marwan, 2011]. However, it may also reflect tangential motion (so-called "sojourn points"), which should be excluded [Marwan *et al.*, 2007]. A diagonal line in a recurrence plot corresponding to (pseudo)periodic



(a) (b)

Fig. 2. (a) RP of the Lorenz oscillator ($\epsilon = 4$, Euclidean norm) and (b) a magnified segment showing that the diagonal lines are compositions of many parallel diagonal lines.

motion should be exactly one point thick, capturing an interval for which trajectories at different points in time remain proximate [Zou *et al.*, 2007]. However, the typical RP construction method causes diagonal lines which exceeded one-point thickness (Fig. 2). Bands of thickness about diagonal line segments constitute redundant information that can introduce artifacts into the line-based measures of recurrence analysis [Zou *et al.*, 2007]. Therefore, the aim is to achieve a balance between (1) not removing relevant dynamical information while (2) minimizing redundant information (keeping diagonal segments thin). This makes the threshold parameter difficult to set in a fully optimized manner.

3. Local Minima Based Thresholding

We now propose an alternative way of thresholding which ensures thin diagonal lines (of almost one point thickness).

For continuous dynamical systems, recurrence behavior with respect to any m-dimensional point $\mathbf{x}_i$ is describable as either moving closer to or further from point $\mathbf{x}_i$. The pattern of decreasing and increasing distances in the distance matrix relative to $\mathbf{x}_i$ will have the form of a relatively smooth function with local maxima and local minima of this distance function denoting points where the direction of motion changes (i.e. diverging or converging to $\mathbf{x}_i$). Hence, the local minima in the distance functions are of particular interest because they correspond to the local closest returns of a segment of the phase space trajectory and will be used to define a recurrence. Over successive time the local minima of the individual distance functions form valleys in the distance matrix parallel to the LOI. These valleys correspond to trajectories that are parallel to one another (even if such trajectories are not proximal).

Since valleys tend to run along lines that are at 45°, the best way to uncover them in the distance matrix is to evaluate the distance profile d_i at time i

$$d_i(\tau) = D_{i+\tau,i-\tau} \quad \text{with}$$
$$\tau = -N + 2\left|i - \frac{N}{2}\right|, \ldots, N - 2\left|i - \frac{N}{2}\right| \tag{4}$$

that runs perpendicular to the LOI (i.e. anti-diagonals of the distance matrix). In addition, this choice of using anti-diagonals maintains the symmetry of the distance matrix. To define a recurrence in the plot we look for local minima in each anti-diagonal of the distance matrix $d_i(\tau)$. Each point found in this manner is qualified as a recurrence point [Fig. 3(b)] and captures the local closest returns (valleys) of the distance matrix. This operation can be performed without choosing any particular ϵ value and theoretically will capture every parallel trajectory present in the signal regardless of the distance between the trajectories. In comparison to standard thresholding, which results in diagonal lines of thickness greater than two [Figs. 3(a) and 3(c)], the local-closest return results in lines of thickness at most two[1] [Figs. 3(b), 3(d) and 4]. This minimizes concern about the redundant information contained in thick lines having a negative effect on recurrence measures.

However, as the requirement for a recurrence is now not one of distance but of "tendency to return", we find more recurrences than using standard definitions of recurrence [Figs. 3(c) and 3(d)]. Depending on the problem, we could use this RP with more recurrences or again introduce a threshold, restricting the recurrences to closer states [Figs. 3(b), 3(d) and 4(b)] in phase space. This constitutes our local minima-based thresholding for RPs. Although we lose the parameter-free approach by again applying a threshold, such an approach can still be beneficial when compared to the standard recurrence definition. For standard thresholding, the recurrence structures (in particular, the diagonal lines) depend on the choice of ϵ, the larger ϵ the more recurrence points will appear in the RP and the thicker the diagonal lines will be. However, for the local closest returns, the recurrence definition is not changing with ϵ in general; the resulting RPs are more independent on variations in ϵ (Fig. 5). In fact, the choice of ϵ only excludes such recurrences that come from phase space trajectories which are far away, but will not change the appearance of diagonal lines (e.g. their thickness). Note that this idea is similar to the perpendicular RPs, where a recurrence is defined for those points which fall into the ϵ-neighborhood and lie in the $(m-1)$-dimensional subspace of the phase space $\mathbb{R}^m$ which

[1] Line thickness of two is due to the fact the odd and even length anti-diagonals alternate and succeeding recurrences thus appear at (i, j) and $(i, j + 1)$ or $(i + 1, j)$.

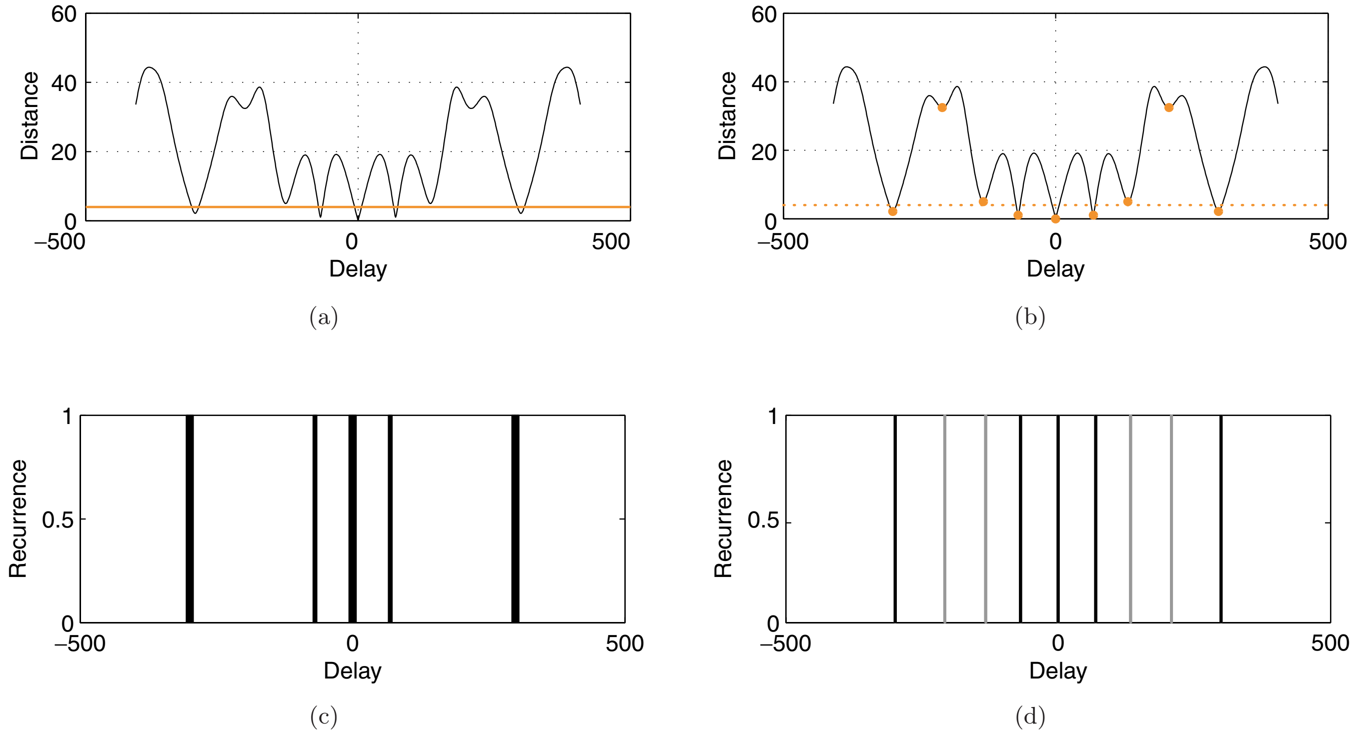


Fig. 3. A distance profile $d_i(\tau)$ through the distance matrix of the Lorenz oscillator. (a) Application of threshold $\epsilon = 4$ (orange line) results in thick lines in the RP (c). (b) Recurrences defined by local minima (orange points). (d) Lines in the RP having mostly thickness one. However, additional recurrences can be found coming from states which are far away in phase space (grey bars). This can be overcome by using additional threshold (red dotted line in (b)).

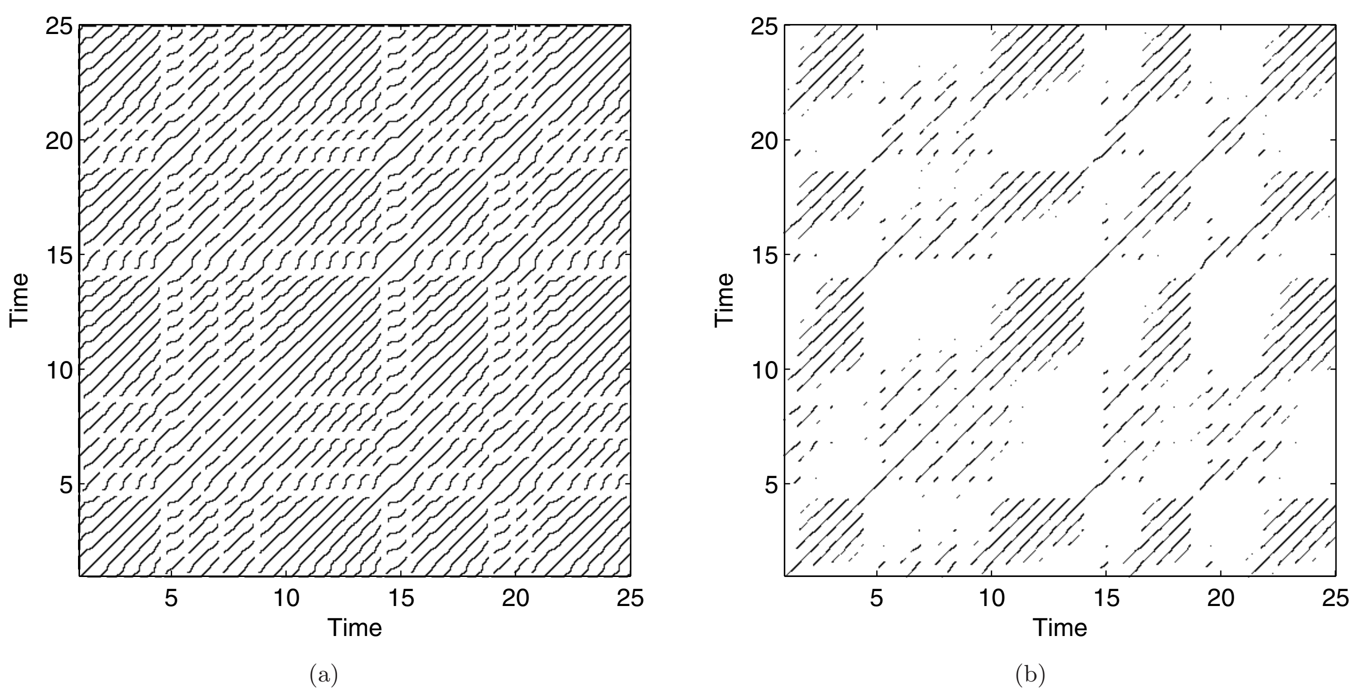
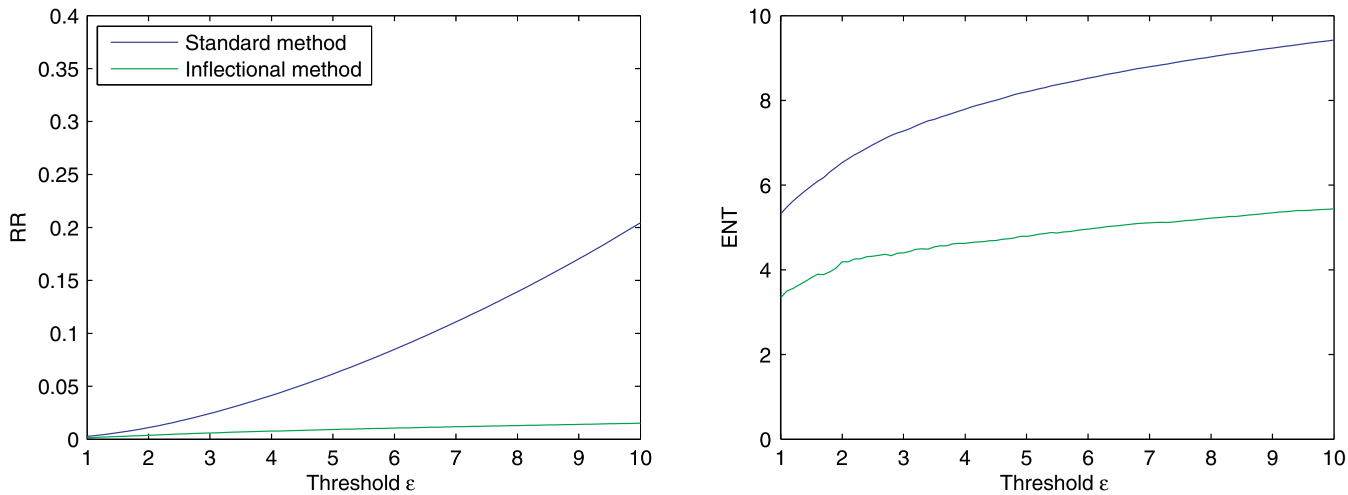


Fig. 4. (a) RP based on local-closest returns for the Lorenz system and (b) applying an additional threshold $\epsilon = 4$.



(a) (b)

Fig. 5. Influence of the threshold ϵ on (a) the number of recurrences as expressed by the recurrence rate RR, and on (b) the distribution of diagonal lines as expressed by the entropy of the length distribution of diagonal lines (ENT) for the example of the Lorenz oscillator. Increasing ϵ caused a larger variation in the recurrence structures for the standard method than for the local minima-based thresholding method.

is perpendicular to the trajectory of this point [Choi *et al.*, 1999].

Line thinness is desired for (pseudo)periodic signals and measures based on diagonal line segments, but interferes with measures based on vertical line segments, as it largely precludes vertical lines in the RP. The local minima-based thresholding method captures the location of the valley but does not retain information about the shape of the valley, which is necessary for the detection of laminar states.

4. Definition for Line Segment

The definition of recurrences in a RP by local minima can cause "distortions" in the diagonal line structures, i.e. lines can appear bowed or with some jumps (Fig. 6). Such distortions can skew diagonal line-related recurrence measures. Therefore, the local minima-based thresholding requires a modification for definitions of line segments. Usually, the definition of a line segment requires sequential recurrence points on diagonals that are parallel to the LOI. Since this definition does not work well with the local minima-based RP thresholding, we propose a slight modification of the definition of a line, allowing some deviations from the 45° direction.

Here a line is conceptualized as any contiguous patch of points in the RP, where the criteria of contiguity specifies that recurrence points can be no more than one sample away in both dimensions of the recurrence matrix (i.e. $R_{i\pm1,j}$ or $R_{i,j\pm1}$). Under the standard method the number of points that make up a line is equal to the number of samples over which the two trajectories remain recurrent, and the period of recurrence will be the same in both dimensions of the distance matrix. Under the new definition of line segments, lines can contain more points that the temporal extents of the line and lines can deviate from 45°. We define the length of such a line as the maximal spatial extent in either x or y-direction. For example, a line that visits points starting from $(50, 30)$ and ending at $(60, 45)$ has an extent of 10 in the x-dimension and of 15 in the y-dimension. The length of this line would then be assigned the value of the larger extent, in this case 15. This definition of line length was chosen to be commensurate with the standard notion that line length is informative of the temporal period over which two trajectories remain recurrent with one another. Other definitions are of course possible.

Besides the new definition of line segments given above, measures that capture nonlinear features of line segments (such as curvatures, slope variation) are also interesting and provide additional information on the system's dynamics [Klimaszewska & Zbebrowski, 2009]. It should also be noted that this new line definition can

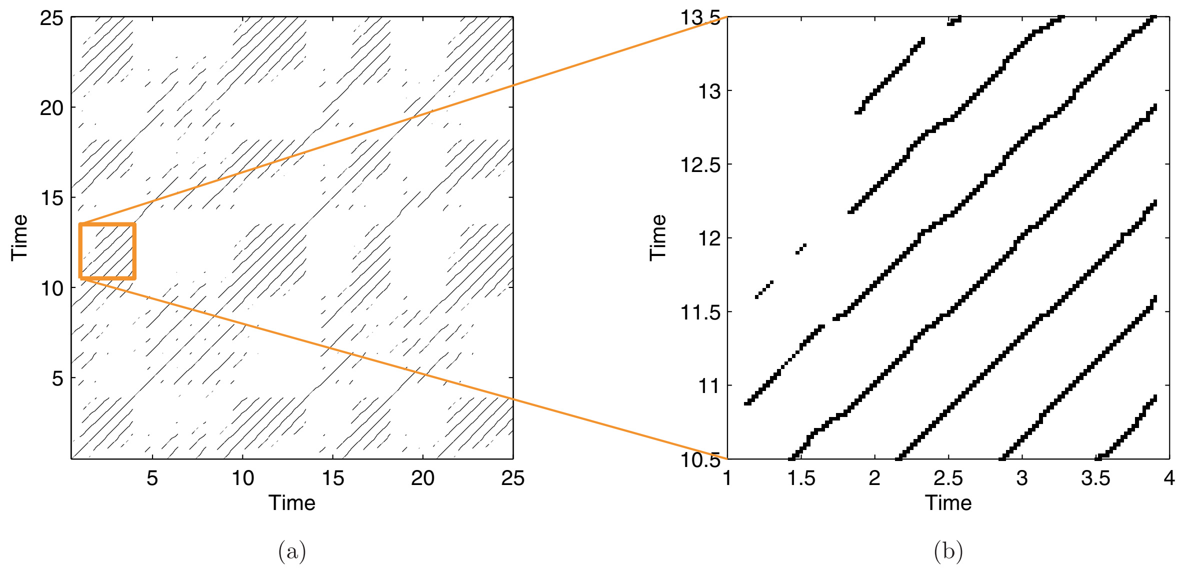


Fig. 6. (a) Local minima-based RP for the Lorenz system and (b) a magnified part of it. Line structures can deviate from being parallel to the LOI.

be used with the standard distance thresholding procedure. Preliminary tests have shown a marked improvement in the sensitivity and accuracy of RQA measures.

5. Efficacy of the Radius-Based and Local Minima-Based Methods

We have presented an alternative way of capturing recurrences in RPs. However, RQA in its current instantiation is already a very powerful and robust tool for studying dynamical properties of time series. For the local minima-based method to be useful it must at least match the efficacy of the standard method in order to show that the benefits it offers do not come with a corresponding cost. To answer this question of efficacy, we have studied dynamical transitions in the Lorenz attractor, Eq. (2), which shows a series of period doubling bifurcations for $\sigma = 10$, $\beta = 8/3$, and ρ varies within the interval [146.7 166.1]. As ρ increases and exceeds a critical value $\rho_0 \approx 166.1$, the system becomes chaotic. We test the efficacy of the local minima method against the standard method for the detection of different dynamical regimes (i.e. in periodic and chaotic regimes). In particular, we have chosen a value very shortly before and after the critical ρ_0, ensuring almost periodic ($\rho = 165.86$) and chaotic dynamics ($\rho = 166.2$).

The Lorenz system has been integrated using the fourth order Runge–Kutta method with an integration step of 0.01 and 10 000 integration steps. The last 2000 iterations of each realization were used in the analysis. The sampling time was 0.01. The test consisted of 1000 realizations of the different parameterizations. Each realization started from random initial conditions in the range of 0 to 1. All three components of the Lorenz system were used for calculation of RPs (i.e. no embedding used).

For both methods, the ϵ value for each run was chosen to be 5% of the standard deviation of the system. The local minima-based method also employed the new definition of a line segment as any contiguous set of recurrence points. The main diagonal was removed for both the standard RP and the local minima-based RP by removing a 10 point Theiler window centered around the LOI.

To test the efficacy, we used an independent samples t-test for each method on each measure to determine if the method was able to statistically discriminate between the two parameter settings. For measures where both methods found statistical significance, we also included Cohen's d as a measure of effect size to compare the statistical power of the two methods. The rule of thumb for Cohen's d is to interpret $d \leq 0.2$ as a small effect, $d \approx 0.5$ as a medium effect and $d \geq 0.8$ as a large effect.

We used four RQA measures in this analysis [Marwan *et al.*, 2007]: determinism DET, maximal diagonal line length $L_{\max}$, mean diagonal line

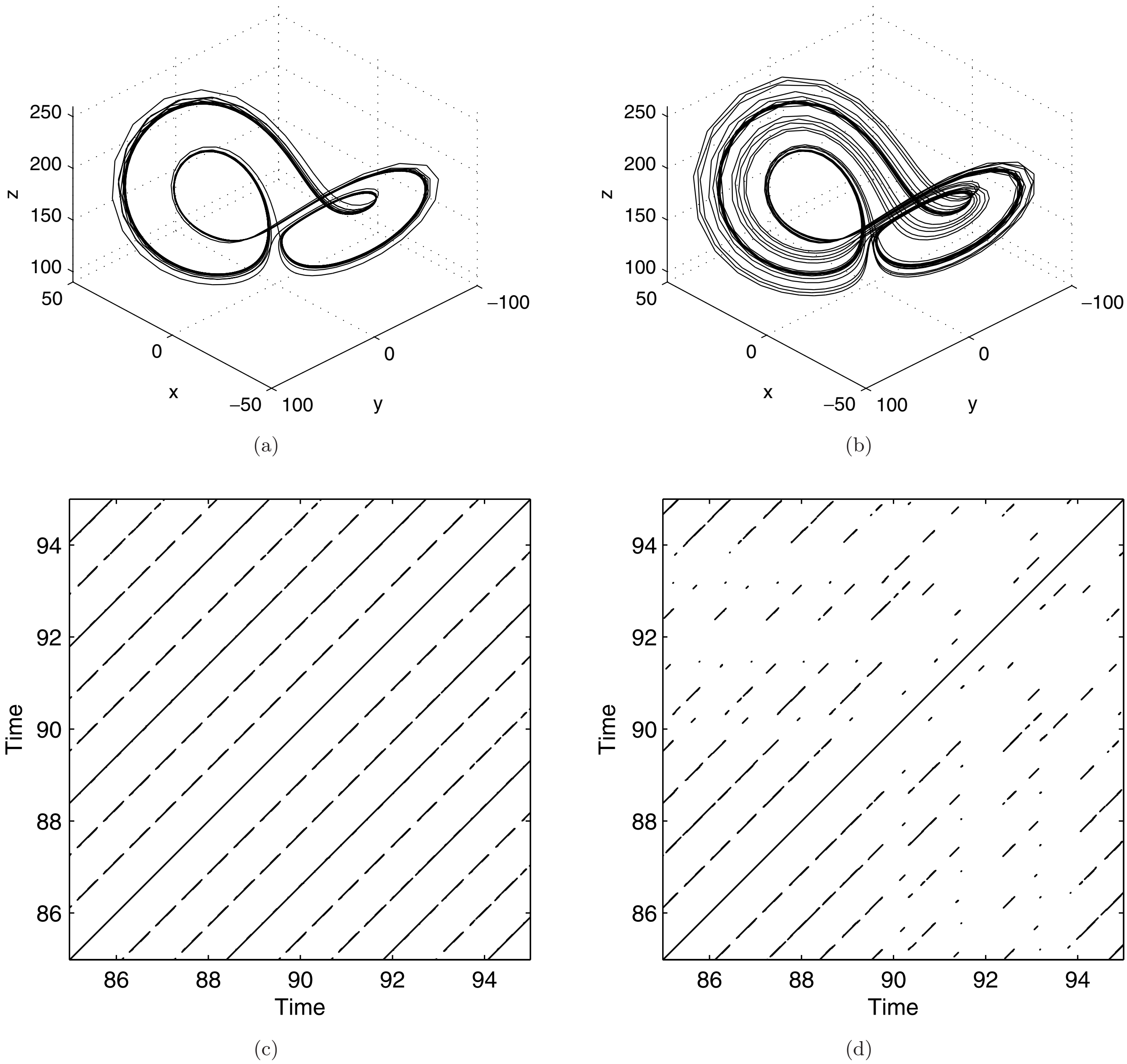


Fig. 7. Phase space portrait of the Lorenz system for (a) $\rho = 165.86$ and (b) $\rho = 166.2$, and corresponding local minima-based RPs for (c) $\rho = 165.86$, and (d) $\rho = 166.2$.

length L, and diagonal line length entropy ENT. We chose this set of measures because they are expected to be affected by the transition from order to chaos and they are based on the diagonal line structures in the RP. We expect that as ρ is increased we should see an increase in ENT due to the increase in chaotic behavior and a decrease in DET, L and $L_{\max}$ due to the growing divergence (i.e. Lyapunov exponent) caused by extending chaotic behavior.

For all four RQA measures, DET, L, $L_{\max}$ and ENT, both methods reveal statistically significant differences (Tables 1 and 2). Both methods show the expected decrease in DET, L and $L_{\max}$ and the increase in ENT for the change of parameters from $\rho = 165.86$ to 166.2. The effect size for the standard method is lower than that for the local minima-based method for all RQA measures, showing that our method has more statistical power.

The results confirm that the local minima-based method matches the performance of the standard method. The effect size is even better using the local minima-based method.

Table 1. Means and standard deviations for the four recurrence measures obtained with the standard method, and corresponding t-values and Cohen's d measure.

	$r = 165.86$ (%)	$r = 166.2$ (%)	t-Value	Cohen's d
DET	99.80 (0.02)	99.76 (0.07)	27.7	0.9
L	21.5 (0.12)	16.2 (1.53)	155	4.9
$L_{\max}$	1661.0 (0)	249.3 (116.0)	544	17.2
ENT	6.55 (0.01)	6.65 (0.18)	24.6	0.8

Table 2. Means and standard deviations for the four recurrence measures obtained with the local minima-based method, and corresponding t-values and Cohen's d measure.

	$r = 165.86$ (%)	$r = 166.2$ (%)	t-Value	Cohen's d
DET	99.99 (0.01)	99.89 (0.07)	67.5	2.1
L	52.5 (0.38)	26.5 (4.87)	238	7.5
$L_{\max}$	1659.0 (0)	252.3 (114.2)	551	17.4
ENT	5.43 (0.01)	5.90 (0.14)	148	4.7

6. Conclusion

We have presented an alternative way of creating a RP by using local minima in the distance matrix. The advantages are that the local minima-based thresholding produces RPs with thin lines (thickness not larger than two), requires less computational effort than perpendicular or iso-directional RPs [Choi *et al.*, 1999; Horai & Aihara, 2002], and the threshold parameter has less impact on the results than it does with the standard method (Fig. 5). Our method shows the same efficacy with the standard method in its ability to differentiate different dynamics by means of RQA. The local minima-based thresholding procedure demonstrates the potential of an unconventional general class of recurrence definitions that directly leverage patterns of distances in the distance matrix to capture dynamic information.

Special emphasis should be placed on the reduced dependency of choosing a distance threshold. While an experienced researcher can navigate the parametric subtleties of recurrence analysis with a high degree of success, a novice to recurrence analysis is likely to be put off by the method or use it incorrectly. As recurrence quantification analysis continues to grow in popularity and breadth of application, making the recurrence analysis more accessible and easier to use will become increasingly more important. Local minima-based thresholding helps this cause by reducing the need to carefully set parameters in order to obtain careful measures.

In this paper, we introduce an alternative way to create recurrence plots, which is particularly useful for continuous dynamical systems. Note that for discrete systems (i.e. maps), we do not suggest using this method. The presented preliminary results provide insights which will be essential for future research. In the following we outline some of the interesting topics:

(1) A systematic comparison of existing RQA measures based on the traditional radius-based thresholding to measures based on our method should be performed. Comparing Fig. 4 to Fig. 1(d), we see that the redundant information represented by the thickness of lines has been removed, which is helpful for reducing the effects of "sojourn" points. With our new definition for line segments, it is obvious to see that the line lengths become longer than they would be using the standard line definition [Fig. 4(a)], however, it remains unknown, to what extent the new line definitions would alter RQA measures. In order to derive consistent results with standard RQA measures, some other alternative definitions may be necessary, e.g. taking into account the information provided by the local curvature of the line segments.
(2) Our initial illustration of using the chaotic Lorenz system can be applied to other continuous systems as well, for instance, a system with high dimensional chaos or contaminated by observational noise. The sensitivity of our method in representing different dynamics and in identifying distinct dynamical transitions is an important area of further work. For example, measures for identifying laminarity, chaos-to-chaos transition etc. would be excellent candidates for testing our method.
(3) Applying an additional threshold ϵ to restrict the local minima of the distance function results in a clean recurrence plot comparable to the results of standard thresholding [Fig. 4(b)]. Dealing with real experimental data, which often contains noise, will require the use of the additional distance threshold. This means we will be faced with the same problem of choosing a threshold value ϵ as when using the traditional method, although as demonstrated there will be a reduced effect of threshold choice on the recurrence measures (Fig. 5). The criterion depending on the signal-to-noise ratio has to be further justified.

(4) Recently, the recurrence matrix Eq. (3) is alternatively quantified by statistical measures from complex network perspectives, i.e. regarding Eq. (3) (modulus the main diagonal) as an equivalent adjacency matrix built from an unweighted undirected network [Marwan *et al.*, 2009; Donner *et al.*, 2010]. In principle, all network measures can be applied directly to the recurrence matrix derived from local minima-based RPs. However, the interrelationship to the corresponding phase space properties should be carefully interpreted. For example in Fig. 4(a), the column-wise recurrence rate is clearly overestimated comparing to Fig. 1(d). Therefore the ratio between the column-wise recurrence rate and the size of the neighborhood is not a good indicator for the local density of the system any more. The clustering coefficient should be interpreted with caution as well.

(5) The performance of the local minima-based RPs with cross recurrence plots [Marwan & Kurths, 2002] or joint recurrence plots [Romano *et al.*, 2004] is still an open question.

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