HOW TO AVOID POTENTIAL PITFALLS IN RECURRENCE PLOT BASED DATA ANALYSIS

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Recurrence plots and recurrence quantification analysis have become popular in the last two decades. Recurrence based methods have on the one hand a deep foundation in the theory of dynamical systems and are on the other hand powerful tools for the investigation of a variety of problems. The increasing interest encompasses the growing risk of misuse and uncritical application of these methods. Therefore, we point out potential problems and pitfalls related to different aspects of the application of recurrence plots and recurrence quantification analysis.

Keywords: Recurrence plot; recurrence quantification analysis; time series analysis; pitfalls.

1. Introduction

Since its introduction in 1987 by Eckmann *et al.* [1987], and the development of different quantification approaches, recurrence plots (RPs) have been widely used for the investigation of complex systems in a variety of different disciplines, as physiology, ecology, finance or earth sciences (e.g. [Marwan, 2008; Schinkel *et al.*, 2007; Zbilut *et al.*, 2004; Facchini *et al.*, 2007; Belaire-Franch, 2004; Pecar, 2003; Trauth *et al.*, 2003; Čermák, 2009]). RPs may attract attention because of their ability to produce beautiful or fancy pictures, as in the case of the colorful representations of fractal sets [Mandelbrot, 1982]. The recent remarkable increase of applications can be traced down in part to several free software packages available for calculating recurrence plots and the corresponding recurrence quantification analysis (RQA). Since these methods are also claimed to be very powerful even for short and non-stationary data, we should be careful not to consider them as a kind of a magic tool, which works on all kinds of data. Owing to the fact that these methods are indeed in some sense powerful and rather adaptable to various problems, it is really important that the user knows how these methods work and has understood the ideas behind the RP and the measures of complexity derived from it. Any uncritical application will lead to serious pitfalls and misinterpretations. As the number of applicants increases, the risk of careless application of RPs and RQA grows.

In this article, we try to highlight some of the pitfalls which can occur during the application of RPs and RQA and present future directions of research for a deep theoretical understanding of the method.

2. Recurrence Plots and Recurrence Quantification

Although similar methods have already existed before, the RP, $R_{i,j} = \Theta(\varepsilon - \|\mathbf{x}_i - \mathbf{x}_j\|)$, for the analysis of the dynamics of a dynamical system by

using its phase space trajectory was introduced by Eckmann *et al.* [1987]. This method can be used in order to visualize the recurrence of a state, i.e. all the times when this state will recur. In the 1990's, a heuristic approach of quantification RPs by its line structures has led to the recurrence quantification analysis (RQA) [Webber Jr. & Zbilut, 1994; Marwan *et al.*, 2002b]. In this approach, the density of recurrence points as well as the histograms $P(l)$ of the lengths l of the diagonal and vertical lines in the RP are quantified. The density of recurrence points (recurrence rate) coincides with the definition of the correlation dimension [Grassberger & Procaccia, 1983]. Moreover, RPs contain much more information about the dynamics of the systems: dynamical invariants like Rényi entropy or correlation dimensions can be derived from the structures in RPs [Faure & Korn, 1998; Thiel *et al.*, 2004], RPs can be used to study synchronization [Romano *et al.*, 2005; Senthilkumar *et al.*, 2006] or to construct surrogate time series [Thiel *et al.*, 2008] and long time series from ensemble measurements [Komalapriya *et al.*, 2010]. For a comprehensive introduction we point to [Marwan *et al.*, 2007].

3. Pitfalls

3.1. *Parameter choice for recurrence analysis*

RP and RQA depend on some parameters which should be properly chosen. For the actual recurrence analysis, a recurrence threshold is necessary. This measure is probably the most crucial one and is discussed in the next subsection.

As already mentioned, the quantification of recurrence structures depends on lines in the RP; by defining a minimal length of such lines, it is possible to adjust the sensitivity of line based recurrence measures. In Secs. 3.3 and 3.4 we will come back to this parameter.

If we start our recurrence analysis from a time series, we have first to reconstruct a phase space by using a proper embedding, e.g. time-delay embedding [Packard *et al.*, 1980]. This involves the proper setting of two additional parameters: the embedding dimension m and the time-delay τ. Although the estimation of dynamical invariants does not depend on the embedding [Thiel *et al.*, 2004], the RQA measures depend on the embedding. Standard

approaches for finding optimal embedding parameters, like false nearest neighbors for embedding dimension and auto-correlation or mutual information for time-delay, can be widely found in the literature (e.g. [Kantz & Schreiber, 1997]). However, it is recommended to visually cross-check the embedding parameters by looking at the resulting RP. Nonoptimal embedding parameters can cause many interruptions of diagonal lines, small blocks, or even diagonal lines perpendicular to the LOI (this corresponds to parallel trajectory segments running in opposite time direction; Fig. 1). The experience has shown that the delay is sometimes overestimated by auto-correlation and mutual information. The embedding dimension has also to be considered with care, as it artificially increases diagonal lines (will be discussed in Sec. 3.3) [Marwan *et al.*, 2007].

In general, it is recommended to study the sensitivity (or robustness) of the results of the recurrence analysis on the parameters (recurrence threshold, embedding parameters).

Although not really a parameter, it is worth to briefly discuss the different recurrence definitions. The most frequently used definition is to consider neighbors in the phase space which are smaller than a threshold value (the recurrence threshold). Distances can be calculated using different norms, like Maximum or Euclidean norm [Marwan *et al.*, 2007]. Maximum norm is sometimes preferred because of its better computational efficiency (only minor differences in the results when compared to Euclidean norm). Another definition of recurrence considers a fixed amount nearest neighbors. This recurrence criterion is used when the number of neighbors is important. Pitfalls related to these recurrence criteria are also discussed in Sec. 3.7. More interesting are combinations of the above criteria with dynamical properties of the phase space trajectory, e.g. perpendicular RPs (Sec. 3.3), or recurrence based on order patterns [Groth, 2005]. Order patterns are representations of the local rank order of a given number d of values of the time series (order pattern dimension). As the number of order patterns is equal to $d!$, the dimension should not be chosen too large, because many order patterns will appear rather seldom and the RP will be sparse. Even $d = 4$ is often already inappropriate, therefore, $d = 3$ is the best choice in most cases (depending on the problem of interest, $d = 2$ may also be appropriate).
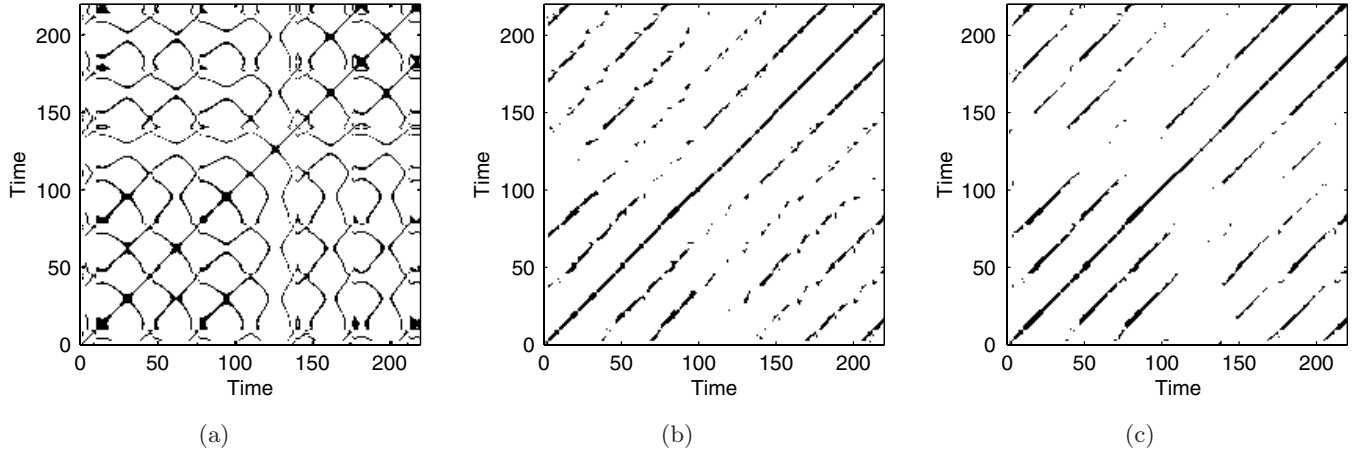
Fig. 1. Recurrence plots of the Rössler oscillator with parameters $a = b = 0.25$ and $c = 40$ using different embedding: (a) $m = 1$, $\tau = 1$, (b) $m = 3$, $\tau = 12$, (c) $m = 3$, $\tau = 6$ (adaptive recurrence threshold to ensure (a) RR = 0.1, (b, c) RR = 0.05). Nonoptimal embedding can cause line structures perpendicular to the main diagonal, wobbly or interrupted lines (a, b).

3.2. *Recurrence threshold selection*

The recurrence threshold ε is a crucial parameter in the RP analysis. Although several works have contributed to this discussion (e.g. [Thiel *et al.*, 2002; Matassini *et al.*, 2002; Marwan *et al.*, 2007; Schinkel *et al.*, 2008]), a general and systematic study on the recurrence threshold selection remains an open task for future work. Nevertheless, recurrence threshold selection is a trade-off from having as small threshold as possible but at the same time a sufficient number of recurrences (i.e. recurrence structures).

However, the diversity of applicability of RP based methods causes a number of different criteria for the selection of the threshold: studying dynamical properties (dynamical invariants, synchronization) requires a very small threshold [Marwan *et al.*, 2007; Donner *et al.*, 2010]; twin surrogates or trajectory reconstruction methods may require larger thresholds [Hirata *et al.*, 2008]; noise corrupted observation data requires even larger thresholds [Thiel *et al.*, 2002]; for studying dynamical transitions, the threshold selection can be even without much importance, because the relative change of the RQA measures does not depend too much on it in a certain range; for the detection of certain signals a specific fraction of the phase space diameter (or standard deviation of the time series) is required [Schinkel *et al.*, 2008].

Several "rules of thumb" for the choice of ε have been advocated in the literature, e.g. a few per cent of the maximum phase space diameter [Mindlin & Gilmore, 1992], a value that should not exceed 10% of the mean or the maximum phase space diameter [Koebbe & Mayer-Kress, 1992; Zbilut & Webber Jr., 1992], or that the recurrence rate RR $= \sum_{i,j} R_{i,j}/N^2$ is approximately 1% [Zbilut *et al.*, 2002]. A recently proposed criterion employing the relationship between recurrence rate and ε defines an optimal value by using the position of the maximum of the first derivative of the recurrence rate $d\mathrm{RR}/d\varepsilon$ [Gao & Jin, 2009]. Such approach can produce ambiguous and highly unstable results, as slight variations in ε (as possible by minor errors in finding this value or by nonstationary time series) cause high variation in the recurrence structure. Next, the position of the maximum of $d\mathrm{RR}/d\varepsilon$ depends strongly on the chosen norm and embedding, and may lead to an overestimation of an optimal ε. And, finally, there are systems which can have more than one maximum [Donner *et al.*, 2010].

Another criterion for the choice of ε takes into account that a measurement of a process is a composition of the real signal and some observational noise with standard deviation σ [Thiel *et al.*, 2002]. In order to get similar results as for the noise-free situation, ε has to be chosen such that it is five times larger than the standard deviation of the observational noise, i.e. $\varepsilon > 5\sigma$. Although this criterion holds for a wide class of processes, it is difficult to estimate the amount of observational noise in the signal.

For (quasi-)periodic processes, it has been suggested to use the diagonal structures within the RP in order to find the optimal ε [Matassini *et al.*, 2002]. In this approach, the density distribution of recurrence points along the diagonals parallel to the

LOI is investigated on dependence of ε in order to minimize the fragmentation and thickness of the diagonal lines with respect to the threshold. However, this choice of ε may not preserve the important distribution of the diagonal lines in the RP if observational noise is present (the estimated threshold can be underestimated).

The selection of an optimal recurrence threshold ε is not straightforward and depends on the particular problem and question.

3.3. *Indicators of determinism*

The length of a diagonal line in the RP corresponds to the time the system evolves very similar as during another time, i.e. a segment of the phase space trajectory runs parallel and within an ε-tube of another segment of the phase space trajectory. Deterministic systems are often characterized by repeated similar state evolution (corresponding to a local predictability), yielding a large number of diagonal lines in the RP. In contrast, systems with independent subsequent values, like white noise, have RPs with mostly single points. Therefore, the fraction of recurrence points forming such diagonal lines (of length $l \geq l_{\min}$)

$$\text{DET} = \frac{\sum_{l \geq l_{\min}} l P(l)}{\sum_{i,j} R_{i,j}} \qquad (1)$$

can be calculated and is, therefore, called *determinism* in the RQA. Somehow this measure can

be interpreted as an indication of determinism in the data. But we should be careful in using the term determinism in a more general or mathematical sense. In a deterministic system, we can calculate the same exact state by using given initial conditions, i.e. there is no stochastic process involved. Different methods can be used to test for determinism in time series, e.g. a combined modeling-surrogate approach [Small & Tse, 2003] or an analysis of the directionality of the phase space trajectory [Kaplan & Glass, 1992].

High values of DET might be an indication of determinism in the studied system, but it is just a necessary condition, not a sufficient one. Even for nondeterministic processes we can find longer diagonal lines in the RP, resulting in increased DET values. For example, the following (nondeterministic) auto-regressive process $x_i = 0.8x_{i-1} + 0.3x_{i-2} - 0.25x_{i-3} + 0.9\xi$ (where ξ is white Gaussian noise) has a DET value of 0.6 (embedding dimension $m = 4$, delay $\tau = 4$, and fixed recurrence rate of 0.1). As it was shown in [Thiel *et al.*, 2003], stochastic processes can have RPs containing longer diagonal lines just by chance (although very rare). Moreover, due to embedding we introduce correlations in the RP and, therefore, also uncorrelated data (e.g. from white noise process) have spurious diagonal lines [Thiel *et al.*, 2006; Marwan *et al.*, 2007] (Fig. 2). Moreover, data pre-processing like low-passfiltering (smoothing) is frequently used. Such pre-processing can also introduce spurious line structures in the RP. Therefore, from just a high value of the RQA measure DET we
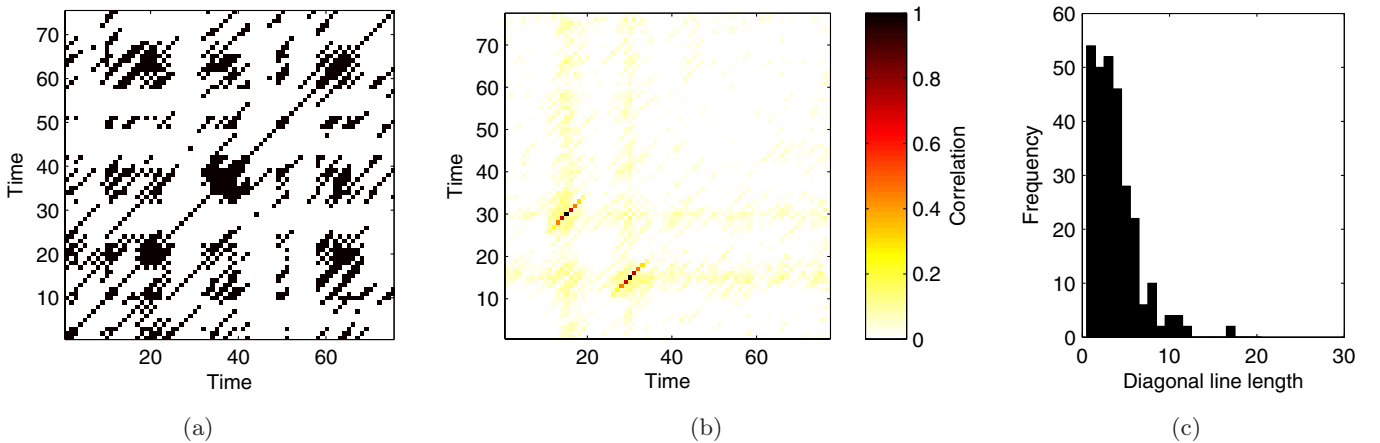

Fig. 2. (a) Recurrence plot of one realization of Gaussian white noise, calculated using embedding dimension $m = 6$, delay $\tau = 1$, and a recurrence threshold of $\varepsilon = 0.2$. The embedding causes a number of long lines. (b) Correlation between a single recurrence point at $(15, 30)$ and other recurrence points in the RP of white noise demonstrating the effect of embedding for a bogus creation of long diagonal lines (estimated from 1000 realizations). (c) The histogram of line lengths found in the RP shown in (a). The maximum length is $L_{\max} = 17$, a value, which would not be uncommon for a deterministic process.

have to be careful in inferring that the studied system would be deterministic. For such conclusion we need at least one further criterion included in the RP: the directionality of the trajectory [Kaplan & Glass, 1992]. One possible solution is to use iso-directional RPs [Horai *et al.*, 2002] or perpendicular RPs [Choi *et al.*, 1999]; if then the measure reaches DET ≈ 1 for a very small recurrence density (i.e. RR < 0.05), the underlying system will be a deterministic one (like a periodic or chaotic system).

3.4. *Indicators of periodic systems*

As explained in the previous section, deterministic systems cause a high value in the RQA measure DET. This measure has been successfully used to detect transitions in the dynamics of complex systems [Trulla *et al.*, 1996]. A frequently used example in order to present this ability is the study of the different dynamical regimes of the logistic map, where DET is able to detect the periodic windows (by values DET $= 1$). Therefore, it is often claimed that this measure is able to detect chaos-period transitions.

However, we can also find such high DET values for non-periodic, but chaotic systems. For example, the Rössler system [Rössler, 1976],

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$
$$= (-y - z, x + 0.25y, 0.25 + z(x - c)), \quad (2)$$

exhibits in the parameter interval $c \in [35, 45]$ a transition from periodic to chaotic states [Fig. 3(a)]. But due to the smooth phase space trajectory and high sampling frequency (sampling time $\Delta t = 0.1$), the RP for the chaotic trajectory consists almost exclusively of diagonal line structures (Fig. 4), resulting in a high value of DET, i.e. DET ≈ 1 [Fig. 3(b)].

A very high value of DET is not a clear or even sufficient indication of a periodic system. High values can be caused by very smooth phase space trajectories. This should also be considered when looking for indications of unstable periodic orbits (UPOs), where DET or mean and maximal line lengths L and $L_{\max}$ may not be sufficient. A solution could be to increase the minimal length $l_{\min}$
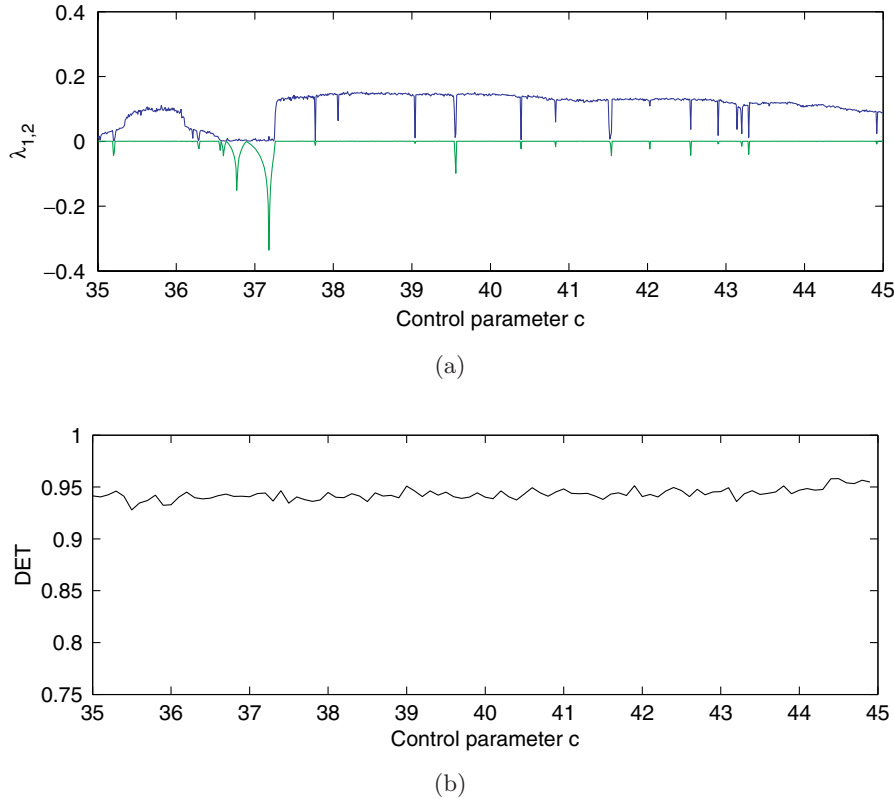

(a)

(b)

Fig. 3.   (a) 1st and 2nd positive Lyapunov exponents of the Rössler oscillator with parameters $a = b = 0.25$ and $c \in [35, 45]$. A periodic window occurs between $c = 36.56$ and $c = 37.25$. However, the DET measures reveals an almost constant very high value of approximately DET $= 0.94$. Used RP parameters: dimension $m = 3$, delay $\tau = 6$, adaptive recurrence threshold to ensure RR $= 0.05$.
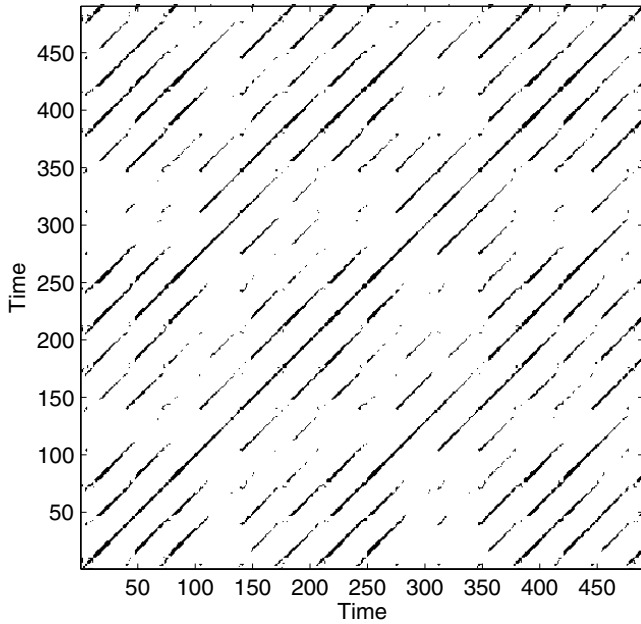
Fig. 4.   Recurrence plot of the Rössler oscillator with parameters $a = b = 0.25$ and $c = 40$. For these parameters, the Rössler system is in a chaotic regime ($\lambda_1 = 0.14$), but the RP consists almost only of diagonal lines. Used RP parameters: dimension $m = 3$, delay $\tau = 6$, adaptive recurrence threshold to ensure RR = 0.05.

of a diagonal recurrence structure which is considered to be a line. However, a better solution is to look at the cumulative distribution of the diagonal line lengths and estimate the K_2 entropy (but this requires much longer time series, cf. Sec. 3.9). Recent work has shown that measures coming from complex network theory, like clustering coefficient, applied to recurrence matrices are more powerful and reliable for the detection of periodic dynamics [Marwan *et al.*, 2009; Zou *et al.*, 2010; Donner *et al.*, 2011].

3.5. *Indicators of chaos*

The RP visualizes the recurrence structure of the considered system (based on the phase space trajectory). The basic idea behind RPs comes, in general, from the study of chaos. Therefore, it can be considered as a nonlinear tool for data analysis. But this cannot be a criterion to understand complex structures in the RP or high values of RQA measures as indicators of chaos or nonlinearity in the dynamical system.

As mentioned above, uncorrelated stochastic systems have mostly short or almost no diagonal

line structures in their RPs, whereas deterministic and regular systems, like periodic processes, have mostly long and continuous diagonal line structures. Chaotic processes have also diagonal, but shorter lines, and can have single recurrence points. Nevertheless, only by looking at the appearance of an RP it is difficult (almost impossible) to infer about the type of dynamics; only periodic and white noise processes can be identified with some certainty.

The alternative is to look at the RQA measures quantifying the structures in an RP which are related to some dynamical characteristics of the system. As diagonal lines in the RP correspond to parallel running trajectory segments, it is clear that the length of these lines is somehow related to the divergence behavior of the dynamical system. Divergence rate of phase space trajectories is measured by the Lyapunov exponent. In fact, the lengths of the diagonal lines are directly related to dynamical invariants as K_2 entropy or D_2 correlation dimension [Faure & Korn, 1998; Thiel *et al.*, 2004]. The K_2 entropy is the lower limit of the sum of the positive Lyapunov exponents.

For example, RQA measures based on the length of the diagonal lines, like determinism DET and mean line length L, also depend on the type of dynamics of the systems (rather low values for uncorrelated stochastic (white noise) systems, higher values for more regular, correlated and also chaotic systems). It has been suggested to measure the length of the longest diagonal line $L_{\max}$ and interpret its inverse DIV = $1/L_{\max}$ as an estimator of the maximal Lyapunov exponent [Trulla *et al.*, 1996]. However, this interpretation incorporates high potential of erroneous conclusions derived from RQA.

First, the main diagonal in the RP (i.e. the line of identity, LOI) is naturally the longest diagonal line, where from it is usually excluded from the analysis. However, due to the tangential motion of the phase space trajectory,[1] subsequent phase space vectors are often also considered as recurrence points (known as sojourn points) [Marwan *et al.*, 2007]. These recurrence points lead to further continuous diagonal lines directly close to the LOI. Without excluding an appropriate corridor along the LOI (the Theiler window), $L_{\max}$ will be artificially large ($\approx N$) and DIV too small.

Second, as explained above, even white noise can have long diagonal lines [Thiel *et al.*, 2003],

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[1]Tangential motion becomes even more crucial and influential for highly sampled or smooth systems.

leading to a small DIV value just by chance (Fig. 2). Although the probability for the occurrence of such long lines is rather small, the probability that lines of length two occur in RPs of stochastic processes is, on the contrary, rather high. Only one line of length two is enough to get a finite value of DIV which might be misinterpreted as a finite Lyapunov exponent and that the system would be chaotic instead stochastic.

Therefore, we have to be careful in interpreting the RQA measures themselves as indicators of chaos. Moreover, such conclusion cannot be drawn by applying a simple surrogate test where the data points are simply shuffled (such a test would only destroy the correlation structure within the data, and, thus, the frequency information).

RP or RQA alone cannot be used to infer nonlinearity from a time series. For this purpose, advanced surrogate techniques are more appropriate [Schreiber & Schmitz, 2000; Rapp *et al.*, 2001].

3.6. *Discrimination analysis and detection of deterministic signals*

RQA is also a powerful tool in order to distinguish between different types of signals, different groups of dynamical regimes, etc. (e.g. [Zbilut *et al.*, 1998; Marwan *et al.*, 2002b; Facchini *et al.*, 2007; Litak *et al.*, 2010]). However, the selection of applicable RQA measures is a crucial task. Not all measures will be useful for all questions. Their application needs justification in terms of the purpose of the intended analysis. For example, for processes which do not contain laminar regimes, or if we are not interested in the detection of such laminar regimes, it would not make sense to use RQA measures based on vertical recurrence structures (like *laminarity* or *trapping time*) [Marwan & Kurths, 2009].

3.7. *Indicators of nonstationarity and transition analysis*

RQA is powerful for the analysis of slight changes and transitions in the dynamics of a complex system. For this purpose we need a time-dependent RQA (a RQA series) that can be realized in two ways (Fig. 5):

(1) The RP is covered with small overlapping windows of size w spreading along the LOI and in which the RQA will be calculated, $R_{i,j}|_{i,j=k}^{k+w-1}$.
(2) The time series (or phase space trajectory) is divided into overlapping segments $x_i|_{i=k}^{k+w-1}$
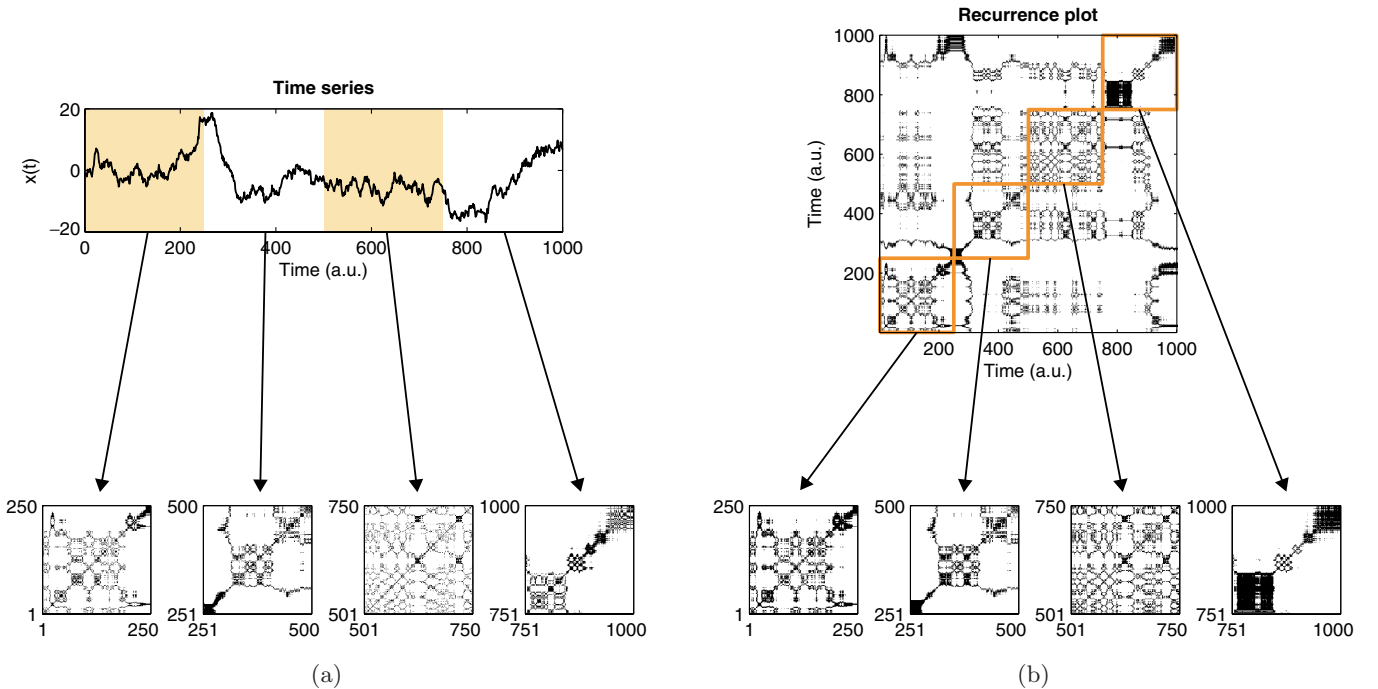

Fig. 5. Two possibilities of windowed RQA: (a) Windowing of time series and (b) windowing of RP. The example is an auto-regressive process: $x_i = 0.95x_{i-1} + 0.05x_{i-2} + 0.9\xi$ (where ξ is white Gaussian noise), the RP is calculated using a constant number of neighbors (10% of all points) and without embedding. The sub-RPs at the bottom clearly demonstrate the differences between the two approaches.

from which RPs and subsequent RQA will be calculated separately.

Such time dependent approach can also be used to analyze the stationarity of the dynamical system.

Here we should note the following important points. The time scale of the RQA values depends on the choice of the point on the window that should be considered as the corresponding time point. Selecting the first point k of the window as the time point of the RQA measures allows us to directly transfer the time scale of the time series to the RQA series. However, the window reaches into the future of the current time point and, thus, the RQA measures represent a state which lies in the future. Variations in the RQA measures can be misinterpreted as early signs of later state transitions (like a prediction). A better choice is therefore to select the center of the window as the current time point of the RQA. Then the RQA considers states in the past and in the future. If strict causality is required (crucial when attempting to detect subtle changes in the dynamics just prior to the onset of dramatic state changes), it might be even useful to select the end point of the window as the current time point of the RQA (using embedding we have to add $(m-1)\tau - 1$). For most applications the center point should be appropriate.

Another important issue that can rise from the different windowing methods (1) or (2), which are only equivalent when we do not normalize the time series (or its pieces) from which the RP is calculated and when we chose a fixed threshold recurrence criterion. If we normalize the time series just before the RP calculation, we get differently normalized segments resulting in different sub-RPs (and thus different RQA results) than such derived directly by moving windows from the RP of the entire time series (Fig. 5 and Table 1). A similar problem arises when we use a fixed number of nearest neighbors for the definition of recurrence, because there is a big difference when considering the entire time series in order to find the k nearest neighbors or just a small piece of it. Nevertheless, both approaches (1) and (2) can be useful and depend on the given question. If we know that the time series shows some nonstationarities or trends which are not of interest, then approaching (2) can help to find transitions that neglect these nonstationarities. But, if we are interested in the detection of the overall changes (e.g. to test for nonstationarity), we should keep the numerical conditions for the entire available time

Table 1. Selected RQA measures derived from (a) windowing of time series and (b) windowing of RP of an auto-regressive process and windowing as shown in Fig. 5.

Window	1–250	251–500	501–750	751–1000
		(a)		
RR	0.10	0.10	0.10	0.10
DET	0.62	0.74	0.48	0.79
L	3.13	3.69	2.75	3.75
		(b)		
RR	0.18	0.12	0.20	0.19
DET	0.81	0.81	0.69	0.95
L	3.78	4.27	2.90	9.50

constant and choose approach (1). Anyway, for each RQA we should explicitly state how the windowing procedure has been performed.

The choice of the window size itself needs the same attention. Because the RQA measures are statistical measures derived from histograms, the window should be large enough to cover a sufficient number of recurrence lines or orbits. A too small window can pretend strong fluctuations in the RQA measures just by weak statistical significance [the RQA measure TREND is very sensitive to the window size and can reveal even contrary results, cp. Fig. 7(b)]. Therefore, conclusions about nonstationarity of the system should be drawn with much care. Moreover, statements on stationarity of the system itself are questionable at all (if not enough knowledge about the system is available), because detected nonstationarity in an observed finite time series does not mean automatic nonstationarity in the underlying system. For example, an auto-regressive process is stationary by definition, but its RP and RQA can reveal a nonstationary signal (Figs. 6 and 7).

3.8. *Significance of RQA measures*

Related to the preceding issue on windowed RQA is the question on the significance of the RQA variation. A suboptimal scaling of the variation of the RQA measures can mislead to conclusions that the studied system has changed its regime or that it would be nonstationary [Figs. 8(a) and 8(b)]. Therefore, it is strongly recommended to cross-check the scaling of the presentation and to present confidence intervals [Figs. 8(c) and 8(d)]. Confidence intervals can be calculated in various ways, but we should
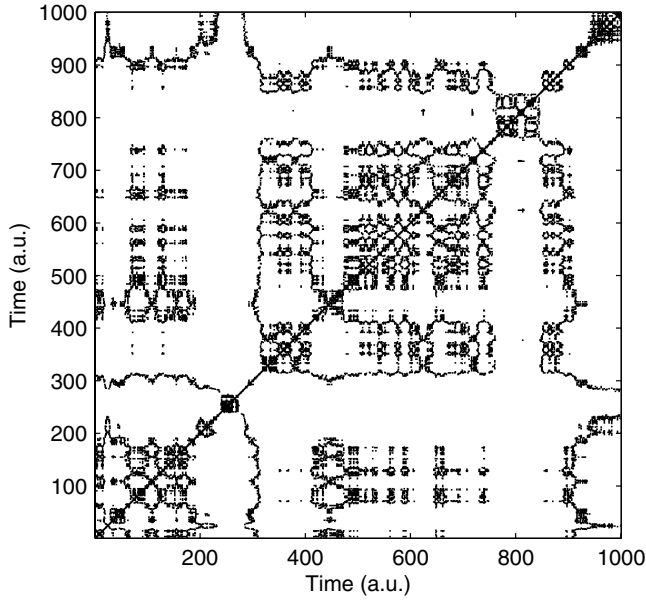
Fig. 6. RP of the same auto-regressive process as presented in Fig. 5, which is by definition stationary. The RP is calculated using maximum norm, $\varepsilon = 2$ and without embedding.

avoid deriving them by simply shuffling the original data. One approach could be a bootstrap resampling of the line structures in the RP [Marwan *et al.*, 2008; Schinkel *et al.*, 2009]. Another approach fits the probability of serial dependences (diagonal lines) to a binomial distribution [Hirata & Aihara, 2011]. Whatever approach we choose, the estimation of the confidence intervals is not a trivial task, but in the future, the standard software for RQA should include such tests.

A common statement on recurrence analysis is that it is useful to analyze short data series. But we have to ask, how short is short? The required length for the estimation of dynamical invariants will be discussed in the following subsection. Applying RQA analysis we should be aware that the RQA measures are statistical measures (like an average) and need some minimal length that a variation can be considered to be significant.

3.9. *Dynamical invariants from short time series*

An RP analysis is appropriate for analyzing short and nonstationary time series, as it is often stated in many reports [Fabretti & Ausloos, 2005; Schinkel *et al.*, 2007; Zbilut *et al.*, 1998]. However, this statement holds actually only for the heuristic measures of complexity as introduced for the RQA or for the detection of differences or transitions in data series. If we are interested in the dynamical invariants derived from RPs, the length N of the time series becomes a more crucial part like it is for the standard methods of nonlinear data analysis.

The derivations of dimensions (D_1, D_2) and dynamical invariants (like K_2) from the RPs hold only in the limit $N \to \infty$ and small ε ($\varepsilon \to 0$).
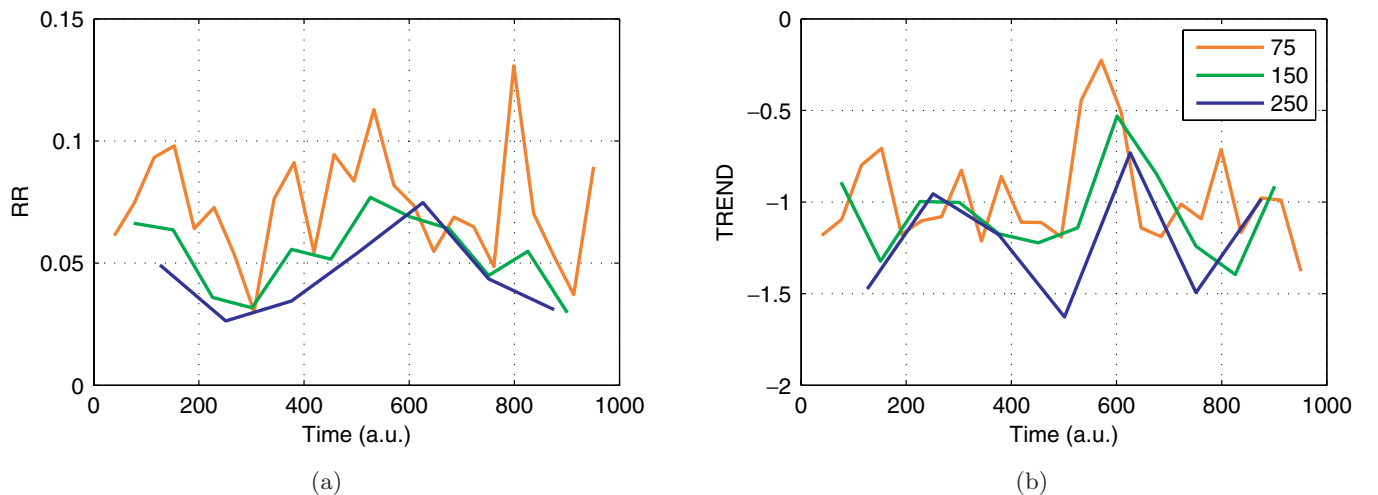

(a)

(b)

Fig. 7. Two exemplary RQA measures, (a) recurrence rate RR and (b) paling trend (TREND), of the auto-regressive process as presented in Fig. 5 for three different window sizes w ($w = 75, 150, 250$). (a) The strong variation in RR pretends a nonstationarity in the signal. (b) TREND depends rather strongly on w, resulting in contrary outcomes, e.g. revealing high values for $w = 75$, but small values for $w = 200$ at the same time period $t = 700, \ldots, 800$. The RQA is calculated using maximum norm, $\varepsilon = 0.3$ and without embedding (the windows are moved by $w/2$, i.e. 50% overlap; the RQA time point is set to the center of the RQA window).
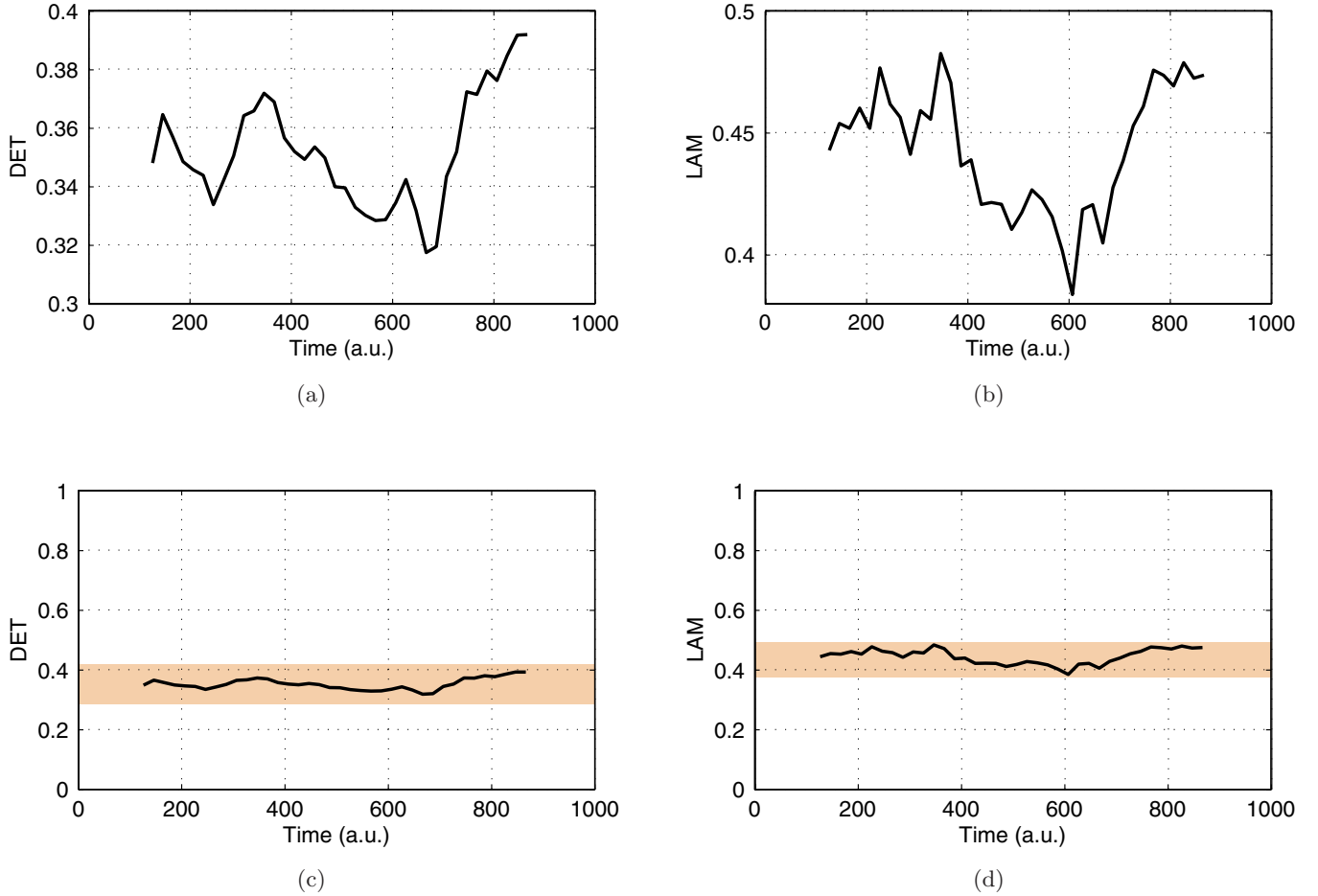
(a)

(b)

(c)

(d)

Fig. 8. Two exemplary RQA measures, (a, c) determinism DET and (b, d) laminarity (LAM), of the auto-regressive process as presented in Fig. 5. (a, b) The scaling of the y-axis is affecting a strong variation in the RQA measures — a potential for wrong conclusions. (c, d) Considering a 5% confidence interval of the RQA measures (details can be found in [Marwan *et al.*, 2008]) and a better value range for the y-axis, we cannot infer that the values of the RQA measures as shown in (a) and (b) significantly vary. The RQA is calculated using a window size of $w = 250$ and a window step of $ws = 20$, using maximum norm, $\varepsilon = 0.3$ and without embedding (the RQA time point is set to the center of the RQA window). LAM is the fraction of recurrence points forming vertical lines in an RP (analogously as DET for the diagonal lines).

Nevertheless, an estimation of dynamical invariants from shorter time series is feasible. We have to regard the following factors if discussing the time series length: the number of orbits representing stretching, the number of recurrences filling out a sufficient part of the attractor, and the number of data points necessary for an acceptable phase space reconstruction [Wolf *et al.*, 1985]. Since these factors may require different minimal lengths, the largest of these lengths should be considered.

For example, numerical considerations for the estimation of the attractor (correlation) dimension D_2 using the Grassberger–Procaccia algorithm [Grassberger & Procaccia, 1983] lead to the requirement $\log N > D_2/2 \log(1/\varrho)$ (where $\varrho = S/\varepsilon$ is the

fraction the recurrence neighborhood of size ε covers the entire phase space of diameter S) [Eckmann & Ruelle, 1998]. Considering a $\varrho = 0.1$ and a decimal logarithm, for finding $D_2 = 10$ we need at least $N = 100\,000$ data points. Furthermore, $\varrho = 0.1$ is actually too large and we need much smaller ε.

For Lyapunov exponents (and analogously for K_2), a rough estimate based on the mentioned requirements suggests minimal time series lengths of 10^{D_2} to 30^{D_2} (with attractor dimension D_2) [Wolf *et al.*, 1985]. Accordingly, a system with $D_2 = 3$ requires 1000–30 000 data points (a more strict consideration even requires $\log N > D_2 \log(1/\varrho)$ [Eckmann & Ruelle, 1998]).

Therefore, to guarantee useful results we need long time series. If we calculate dimensions or K_2

from short time series the results are probably worthless.

3.10. *Synchronization and line of synchronization*

Cross recurrence plots (CRPs) can be used for the investigation of the simultaneous evolution of two different phase space trajectories [Marwan *et al.*, 2002a; Marwan & Kurths, 2002; Zolotova *et al.*, 2009; Ihrke *et al.*, 2009]. The *line of identity* (LOI) in the RP becomes a *line of synchronization* (LOS) in the CRP. Two more-or-less identical systems but with differences on the time-scale will reveal a bowed LOS [Marwan *et al.*, 2002a; Marwan & Kurths, 2005]. An off-set of the LOS away from the main diagonal is an indication of a phase shift or a delay between the two considered systems.

However, because this method tests if the two trajectories visit the same region in the phase space, it can be used only to study complete synchronization (CS) or a kind of a generalized correlation (although with possible delays), or to get the relation between the transformations between their time-scales. Moreover, the data under consideration should be from the same (or a very

comparable) process and, actually, should represent the same observable. Therefore, the reconstructed phase space should be the same.

For the study of the LOS the distance matrix may be more appropriate because it contains more information, especially if the data series show non-stationarities. Then, the LOS can be found by using efficient algorithms like dynamic time warping [Sakoe & Chiba, 1978]. Nevertheless, it is always very important to check if the found LOS makes sense; for instance, it is possible to find several LOS (cp. Application in magneto-stratigraphy in [Marwan *et al.*, 2007]).

3.11. *Macrostructures and sampling*

For the visual interpretation of an RP and also for a reliable RQA we should remember that our data are discretised time or data series. The sampling of the signal has an importance which should not be underestimated. If the sampling frequency is just one magnitude higher than the system's main frequencies, and their ratio is not a multiple of an integer (i.e. we have an intrinsic phase error), an interference triggered by the sampling of the continuous
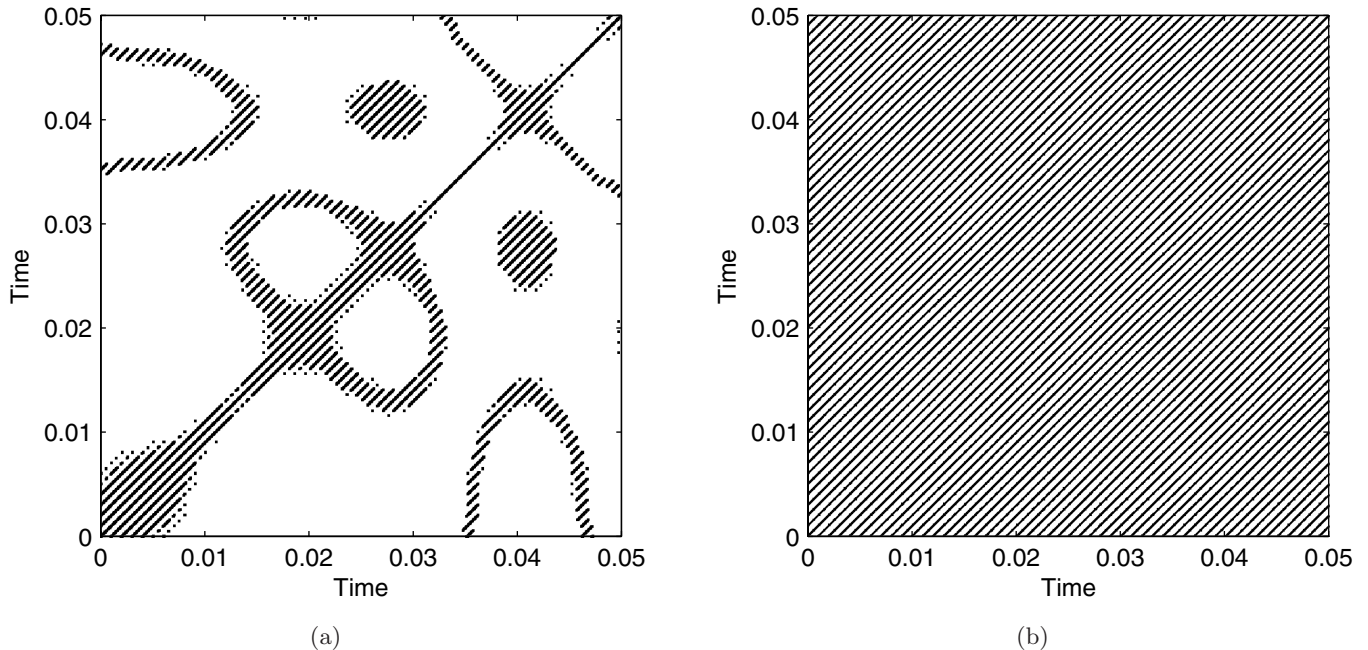

(a) (b)

Fig. 9. RP of a modulated harmonic oscillation $\sin(2\pi 1000(\pi + t) + 2\pi \sin(2\pi 44t)t)$. (a) Nontrivial macrostructures (gaps) in the RP due to the interference of the sampling frequency of 1 kHz and the frequency of the modulated harmonic signal. (b) Corresponding RP as shown in (a), but for a higher sampling frequency of 10 kHz. As expected, the entire RP now consists of the periodic line structures due to the oscillation. Used RP parameters: dimension $m = 3$, delay $\tau = 1$, recurrence threshold $\varepsilon = 0.05\sigma$ (maximum norm).
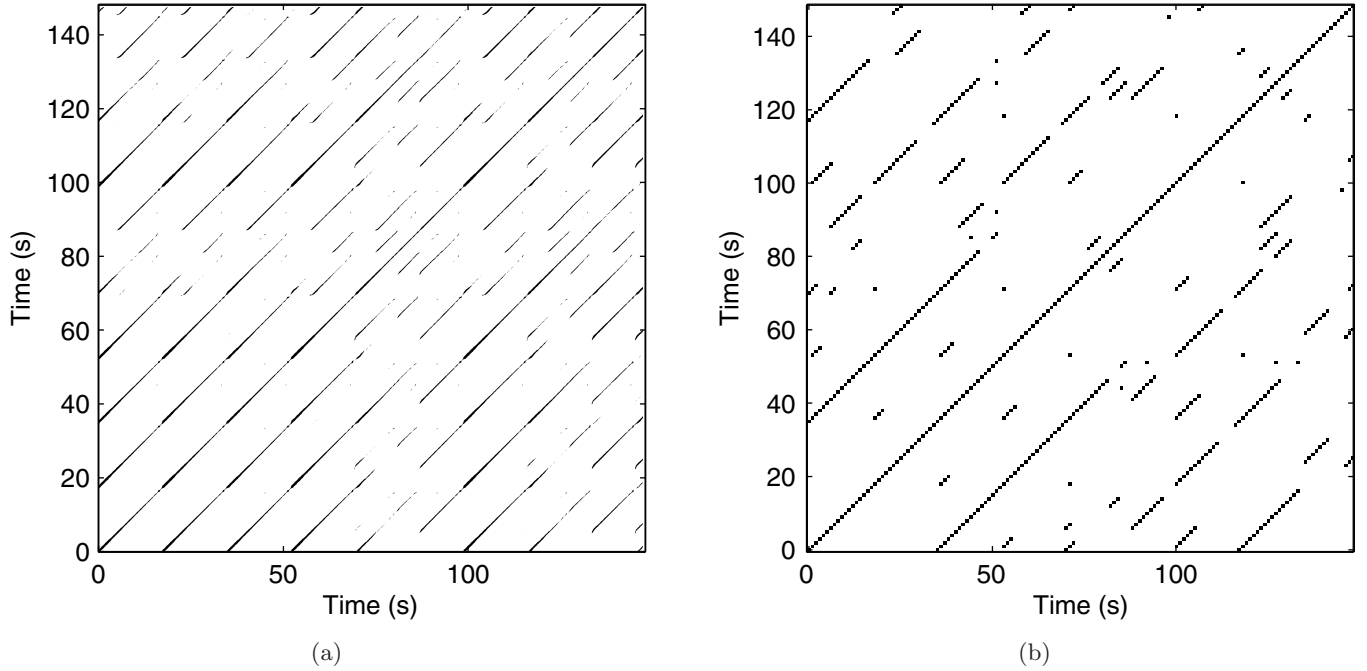
(a)

(b)

Fig. 10.   RP of the x-component of the Rössler oscillator, Eq. (2), with parameters $a = b = 0.2, c = 5.7$. The sampling time is (a) $\Delta t = 0.05$ and (b) $\Delta t = 1$. The embedding was chosen in both settings to be equivalent: dimension in (a) and (b) is $m = 3$, the delay in (a) $\tau = 20$, but in (b) $\tau = 1$; recurrence threshold $\varepsilon = 1.5$ (maximum norm). Due to the low sampling in (b), many diagonal lines vanish.

signal can produce large empty regions in the recurrence matrix, although they should not be there [Facchini *et al.*, 2005; Facchini & Kantz, 2007]. Nonstationarities or modulations in frequency or phase cause nontrivial gaps or macrostructures in the recurrence matrix (Fig. 9). We should be aware that such gaps can occur in particular when we use a low sampling frequency. The recurrence structure of
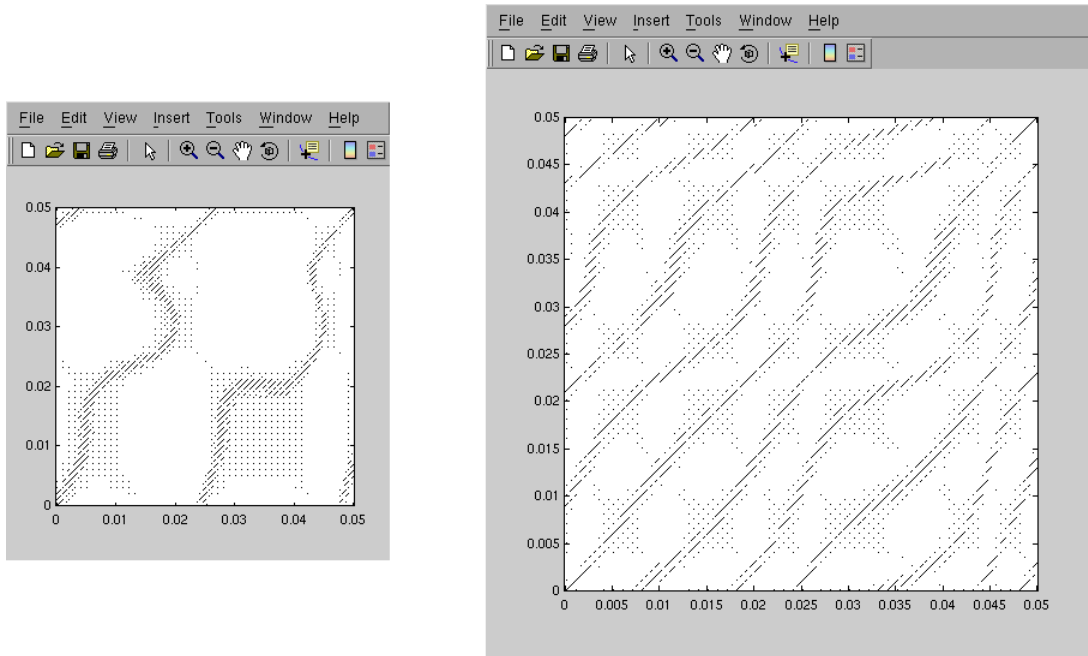

Fig. 11.   Screenshot of the RP as shown in Fig. 9(b) for two different window sizes of display on a computer screen (using Matlab$^{\circledR}$). Although the RP consists only of continuous diagonal lines as represented in Fig. 9(b), its size ($N = 5511$) exceeds the screen resolution and requires downsampling, leading to artifical macrostructures.

interest can appear rather different; diagonal lines can vanish or can be reduced to just single points yielding biased RQA measures (Fig. 10).

Nevertheless, tiny modulations in frequency or phase in oscillating signals can be detected by RPs, which are nondetectable by standard methods (spectral or wavelet analysis). This turns RPs to a powerful tool for the analysis of slight modulations in oscillatory signals like audio signals.

Please note that macrostructures are also an apparent problem when displaying large RPs on a computer screen (and up to a certain amount on print outs). The resolution of modern computer screens is around 72 ppi (points per inch, 72 ppi corresponds to around 28 points per centimetre). The presentation of RPs in a window of, e.g. 6 inch allows only the display of around 430 points. Larger RPs will be rendered using downsampling or interpolation, resulting in similar interference effects and artificial secondary macrostructures as described above; such macrostructures will even change for different window sizes (Fig. 11). Therefore, we should take care in visual interpretation of patterns found in large RPs which are represented on computer screens.

4. Conclusions

We have illustrated several problems regarding the application of recurrence plots (RPs) and recurrence quantification analysis (RQA) which need our attention in order to avoid wrong results. The uncritical application of these methods can lead to serious pitfalls. Therefore, it is important to understand the basic principles and ideas behind the measures of complexity forming the RQA and the different techniques to study the numerous phenomena of complex systems, like transitions, synchronization, etc. Nevertheless, the recurrence plot based techniques are still a rather young field in nonlinear time series analysis, and many open questions remain. For example, systematic research is necessary to define reliable criteria for the selection of the recurrence threshold, and the estimation of the confidence of the RQA measures will be a hot topic in the near future.

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