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Scaling behaviour for recurrence-based measures at the edge of chaos

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Abstract – The study of phase transitions with critical exponents has helped to understand fundamental physical mechanisms. Dynamical systems which go to chaos via period doublings show an equivalent behavior during transitions between different dynamical regimes that can be expressed by critical exponents, known as the Huberman-Rudnick scaling law. This universal law is well studied, *e.g.*, with respect to the Lyapunov exponents. Recurrence plots and related recurrence quantification analysis are popular tools to investigate the regime transitions in dynamical systems. However, the measures are mostly heuristically defined and lack clear theoretical justification. In this letter we link a selection of these heuristical measures with theory by numerically studying their scaling behavior when approaching a phase transition point. We find a promising similarity between the critical exponents to those of the Huberman-Rudnick scaling law, suggesting that the considered measures are able to indicate dynamical phase transition even from the theoretical point of view.

Introduction. – Many dissipative dynamical systems, $\boldsymbol{f}(\boldsymbol{x}, a)$, where $\boldsymbol{x}$ is the state vector and a is a control parameter, possess complex structures in some specific regions of their own phase spaces, such as the logistic map [1], the cubic map [2], the Chirikov map [3], forced pendulum [4], and the Rayleigh-Benard system in a box [5]. As a common property, these kinds of systems exhibit transitions from a periodic regime to a chaotic one via period doubling. The period doubling bifurcations are not only a fundamental feature in prototypical models, it has been found in Nature and experimental setups as well. Examples include collection of cardiac cells [6,7], semiconductors [8] and RLC circuits [9]. Generally the behaviour of a dynamical system is characterized by the maximum Lyapunov exponent $\lambda_{\max}$ which is an index that characterizes the rate of divergence of infinitesimally close trajectories while the system evolves in time. During the transition, the scaling behaviour of the Lyapunov exponent is a crucial property to identify the onset of chaos. Huberman and Rudnick theoretically studied the scaling of the Lyapunov exponent of one-dimensional systems in the vicinity of the critical phase transition point for a set of the control parameter a. Note that, there is only one Lyapunov exponent for one-dimensional systems, therefore $\lambda = \lambda_{\max}$. They found that the Lyapunov exponent scales with an exponent ν: $\lambda \propto (a - a_c)^\nu$, where a_c is the critical control parameter where the regime transition occurs [10].

The Huberman-Rudnick scaling formulation is very similar to the magnetization's one: $M \propto |T - T_c|^\gamma$ near the second-order critical phase transition point (T_c) of magnetic systems [11]. Therefore, as similar as in the scaling of the magnetization M, the scaling relation between different dynamical regimes implies that the Lyapunov exponent λ is such an order parameter describing the order (disorder) of the systems and exhibiting their criticality [12]. In the theory of critical phenomena, many similar formulations of scaling laws can be observed such as the liquid-gas density difference, correlation functions, etc. [13].

Recurrence is one of the fundamental features of dynamical systems. Introduced by Poincaré in 1890, the Poincaré recurrence theorem states that almost all trajectories of dynamical systems will return very close to their previous positions after a sufficiently long but finite time [14]. A flow map is a dynamical system defined by a set of ordinary differential equations. If the phase space of a flow map has a bounded volume, then the Poincaré recurrence theorem is always valid [15].

Among the different approaches of investigating dynamical properties by recurrences, the recurrence plot (RP) [16] and derived quantification techniques are powerful nonlinear tools to analyze different aspects of dynamical systems [17]. In order to understand the underlaying dynamics of complex systems, this method has been applied to various real-world systems in neuroscience [18], financial science [19], geophysical [20] and climate systems [21] as well as on low- and high-dimensional model systems [16,22–24].

The complexity measures in the RP framework, known as "recurrence quantification analysis" (RQA), are based on point density and line structures visible in the RP, and provide an alternative for quantifying order and disorder of physical systems. In order to show robustness, stability, and effectiveness of RPs, it is very important to verify that the RQA measures have a unique scaling behaviour and are related to present universal scaling laws.

In this letter, we will uncover unique scaling exponents of the RQA measures for systems with transitions between different dynamical regimes via period doubling. In particular, we will analyze the discrete logistic map and the continuous Rössler oscillator by using band regions at the edge of chaos. We will investigate the invariance of the scaling exponents for selected RQA measures and their relationship to the Huberman-Rudnick scaling exponent and, thus, the Lyapunov exponent.

Huberman-Rudnick scaling law. – Due to the self-similarity of any dynamical systems which possess transitions via period doublings to chaos, the scaling relation for period doubling in a range of the control parameter a is given by

$$|a - a_c| \sim \delta^{-n}, \tag{1}$$

where $\delta = 4.669\ldots$ is the Feigenbaum constant [25,26]. Equation (1) enables us to localize the control parameter value at the bifurcation from the 2^n period to 2^{n+1}. On the other hand, it is possible to obtain eq. (1) from the Huberman-Rudnick scaling law analytically. This theoretical transformation shows that the envelope of the Lyapunov exponent exhibits a universal scaling behaviour, like an order parameter close to the critical point of an abrupt phase transition [11]. The Huberman-Rudnick relation is given by

$$\lambda = \lambda_0\, (a - a_c)^{\nu}, \tag{2}$$

where $a > a_c$, $\nu = \ln 2/\ln\delta \approx 0.45$, λ is the Lyapunov exponent, and λ_0 is a constant. It is well known that for a slightly larger than the critical a_c (chaos edge), there exist 2^n ($n = 1, 2, \ldots, \infty$) chaotic bands [10]. In that region ($a > a_c$), assume we initialize the system from two different initial conditions in the same band and separated by a distance d_0. After $N^* = 2^n$ iterations, the trajectories will be back in the initial band, since they started from the same one, where N^* is the minimum time step that is necessary to determine the unique band structure. At the same time, regarding the chaotic regime behaviour, the trajectories diverge from each other exponentially fast. Thus, we can estimate the new distance after 2^n iterations by $d_{2^n} = d_0 e^{\lambda 2^n} = d_0 e^{\lambda_0}$, where $\lambda_0 = \lambda 2^n$ is the effective Lyapunov exponent (a constant value) [10]. Substituting the effective Lyapunov exponent into eq. (2) gives

$$2^{-n} = (a - a_c)^{\ln 2/\ln\delta}. \tag{3}$$

It is clearly shown that the distance between a and a_c depends on the number of chaotic bands (2^n).

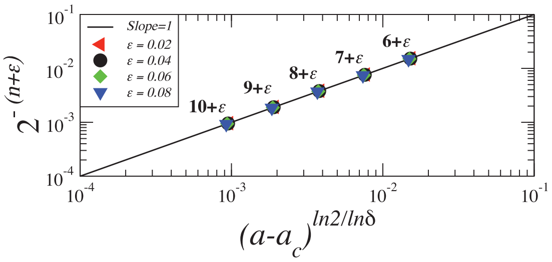

Fig. 1: (Colour on-line) While the threshold value of the recurrence plot ε approaches to the transition point of the logistic map a_c with $(n+\varepsilon, a)$-tuples, the scaling of the $2^{(n+\varepsilon)}$ with $(a-a_c)^{\ln 2/\ln\delta}$ gives a line which has a constant slope, *Slope* = 1. Also, it is shown that the scaling relation in eq. (2) is robust for different values of ε with this transformation. Numerical observations are expected in the figure, all $(n+\varepsilon, a)$-tuples with different ε are on the same line with the slope equal to 1 since the effective Lyapunov exponent λ_0 equals $\lambda 2^{(n+\varepsilon)}$.

Recurrence plot. – In a given m-dimensional phase space, if the states of two points are sufficiently close to each other, they are considered as recurrent states. Formally, for a given trajectory $\boldsymbol{x}_i$ ($i = 1, 2, \ldots, N, \boldsymbol{x} \in \mathbb{R}^m$) where N is the trajectory length, the recurrence matrix is defined by $R_{i,j}(\epsilon) = \Theta\,(\epsilon - \|\boldsymbol{x}_i - \boldsymbol{x}_j\|)$, where ϵ is the neighbourhood threshold, $\|\cdot\|$ is the Euclidean norm, and $\Theta(x)$ is the Heaviside step function [17]. If only a one-dimensional time series is given, time-delay embedding can be used to reconstruct the trajectory phase space for a time series $\{u_i\}_{i=1}^N$ [27], $\boldsymbol{x}_i = (u_i, u_{i+\tau}, \ldots, u_{i+(m-1)\tau})$, where m is the embedding dimension and τ is the embedding delay. For the consistency of both applications in this letter, although an embedding is not necessary for one-dimensional maps (*i.e.*, $m = 1$), we have used both an embedding of $m = 3$, $\tau = 1$ and threshold distance $\epsilon = 0.1\ d$, where d is the maximal phase space diameter of the trajectory in accordance with results in [17,23].

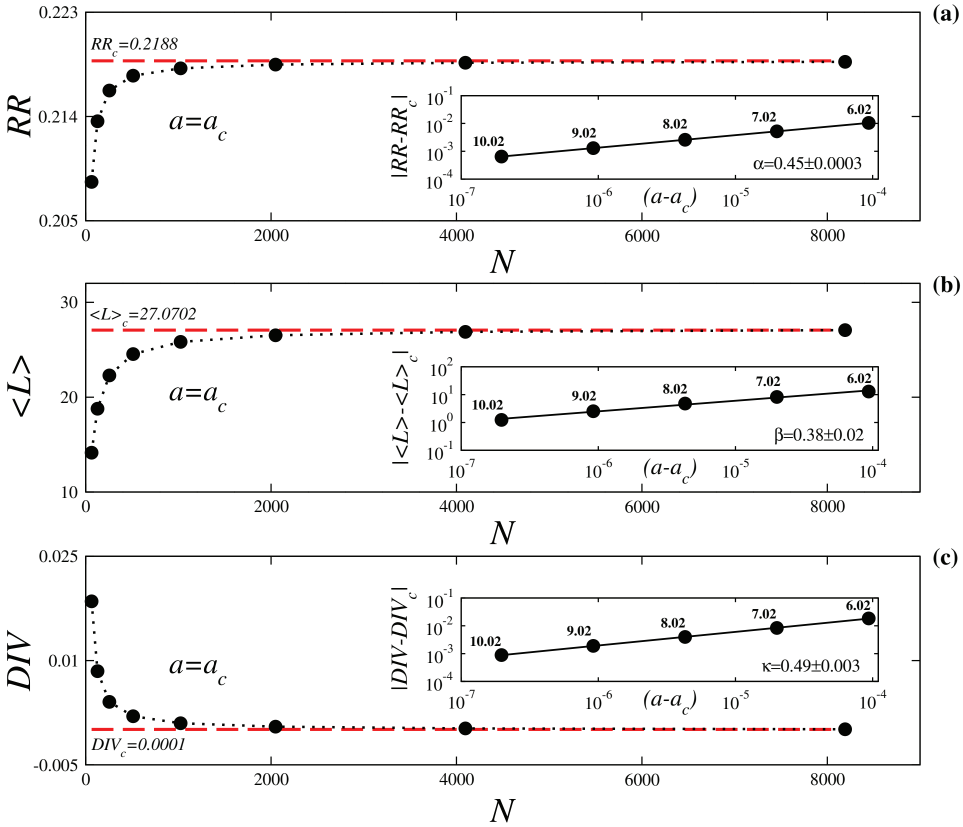

Fig. 2: (Colour on-line) The saturation and the scaling behaviours of RQA measures: (a) the recurrence rate RR, (b) the average diagonal length $\langle L\rangle$ and (c) just scaling of divergence DIV, for the logistic map.

In this work, we use the point-density–based RQA measure, recurrence rate (RR), and two diagonal-structure–based RQA measures, the average diagonal line length $\langle L\rangle$ and the divergence DIV [17]. RR is defined by the mean of all elements in the RP,

$$RR = \frac{1}{N^2}\sum_{i,j=1}^{N} R_{i,j} \tag{4}$$

and the average diagonal line length $\langle L\rangle$ by

$$\langle L\rangle = \frac{\sum_{l=l_{min}}^{N} lP(l)}{\sum_{l=l_{min}}^{N} P(l)}, \tag{5}$$

where $P(l)$ is the histogram of the diagonal structures in the RP. The longest diagonal structure in RP, $L_{\max}$, is defined as $L_{\max} = \max(\{l_i|P(l_i)>0; i=1,2,\ldots\})$, and its inverse

$$DIV = 1/L_{\max}, \tag{6}$$

is related to the divergence behavior of the phase space trajectories [17].

As mentioned before, recurrence is a fundamental characteristic of dynamical systems and recurrence-based measures are, therefore, promising candidates for studying the relation between the Lyapunov exponent and the control parameters. Therefore, it is expected that scaling behaviour exists for recurrence-based measures similar to the Huberman-Rudnick universal scaling law.

The logistic map. – In order to demonstrate the scaling behaviour of RQA measures, we consider a well-known one-dimensional discrete map, called the logistic map, defined as

$$x_{t+1} = 1 - a\,x_t^2, \tag{7}$$

where x_t is a real number in the range $[-1,1]$ and a is the control parameter in the range $(0,2]$. Moreover, changing a causes a transition from the periodic to the chaotic regime at the critical control parameter $a_c = 1.401155189\ldots$. Approaching a_c from the left-hand side (from the periodic region), period doublings occur up to 2^∞ periods at $a=a_c$. In the other direction, approaching a_c from the right-hand side (from the chaotic region), changing a causes band splitting up to an infinite number of bands which split up at a_c [28]. It is worth noting that there are reverse bifurcations of 2^{n+1} bands merging into 2^n bands in the chaotic region ($a>a_c$).

The time series created from the logistic map possess long-range correlations leading to q-Gaussian distributions of sums of iterations and a fractal structure in the vicinity of the chaos threshold [29]; it was recently shown that the correlation length, the box-counting fractal dimension

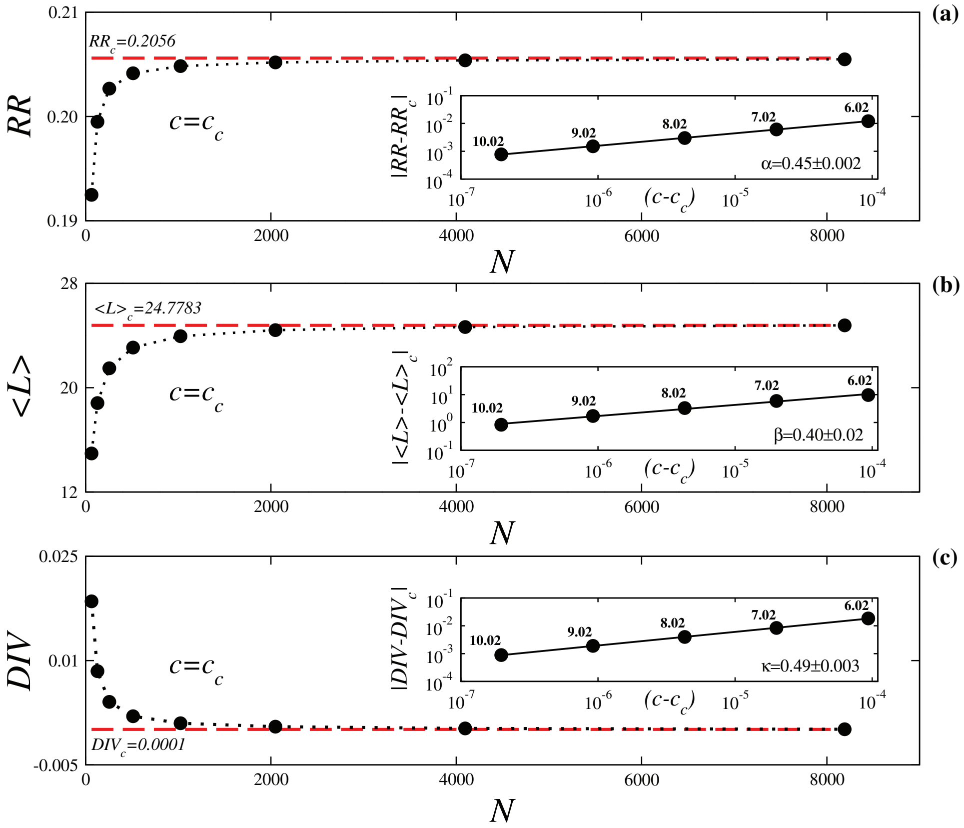

Fig. 3: (Colour on-line) The saturation and the scaling behaviours of RQA measures: (a) the recurrence rate RR, (b) the average diagonal length $\langle L\rangle$ and (c) just scaling of divergence DIV, for the Rössler oscillator.

and the Lyapunov exponent have a special criticality corresponding to a power law scaling with the same exponent ($|\nu| \approx 0.45$) as approaching to the chaos threshold within the Huberman-Rudnick universal scaling law [30,31].

In order to numerically obtain the scaling relation among the RQA measures and the distance between the control parameter and its critical value ($(a-a_c)$), we need to know the critical values of the RQA measures at a_c. However, to localize a_c with infinite precision is not the only ingredient to attain the critical value of these measures. It is also necessary to take $N \to \infty$. On the other hand, in numerical experiments, neither the precision of a_c nor N can approach infinity. But we can detect the critical value with asymptotically saturation behavior of the RQA measures with increasing trajectory length N for a given a_c with very high precision and after discarding long transients $N = 2^{12}$.

To avoid numerical errors in our simulations, we will take n values as the $n \to n+\varepsilon$ transformation, where ε is a very small arbitrary number [29,30,32]. Although the scaling is robust for various ε values (fig. 1), it is important that arbitrarily selected ε should be fixed for all different control parameters a, since different values of ε can cause the system to take a different path to chaos. The slope of the straight line is one and arises from the constant effective Lyapunov exponent $\lambda_0 = \lambda 2^{n+\varepsilon}$ according to eq. (3). From now on, we select $\varepsilon = 0.02$ for all our numerical simulations in this paper.

Now we estimate the critical values RR_c, $\langle L\rangle_c$ and DIV_c. Note that the Lyapunov exponent λ and divergence DIV can vanish at a_c for only $N \to \infty$ ($\lambda, DIV \to 0$). For increasing time series length N, at a_c the RQA measures approach a critical value, estimated as $RR_c = 0.2188$, $\langle L\rangle_c = 27.0702$ and $DIV_c = 0.0001$ (figs. 2(a), (b), (c), respectively). The corresponding scaling exponents are similar to Huberman-Rudnick's exponent $\nu \approx 0.45$ while approaching to the critical threshold with $(n+\varepsilon, a)$-tuples of N^* iterations (fig. 2). For the recurrence rate, $|RR - RR_c|$ scales with $(a-a_c)^\alpha$, where $\alpha = 0.45 \pm 0.0003$. The scaling for $\langle L\rangle$, *i.e.*, $|\langle L\rangle - \langle L\rangle_c| \propto (a-a_c)^\beta$, has the critical exponent $\beta = 0.38 \pm 0.02$ and the scaling for the divergence, *i.e.*, $|DIV - DIV_c| \propto (a-a_c)^\kappa$, and we find the critical exponent $\kappa = 0.49 \pm 0.003$.

The Rössler attractor. – In order to show the generality of our work, we consider now a continuous system, the Rössler oscillator,

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right) = (-y-z, x+ay, b+zc), \tag{8}$$

where c is the control parameter, while fixing the parameters $a = b = 0.2$ [33]. We need to reduce the dimension

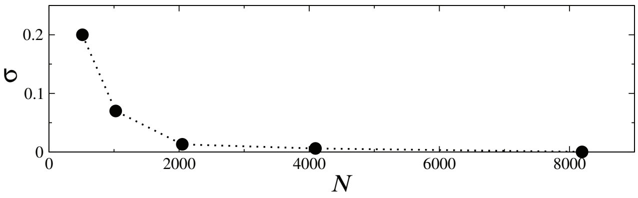

Fig. 4: (Colour on-line) The convergence behaviour of the error described by the standard deviation of the slope ($\pm\sigma$) for the RR exponent α while N increases.

of the Rössler system, *e.g.*, by applying the Poincaré section method to its x-component. By applying this approach to the Rössler system, it has been shown that the period doubling leads to chaos [34]. Using the Poincaré section method, the unique scaling could be studied even in higher-dimensional systems. This property satisfies our requirement for the unique scaling. The critical value of the control parameter is $c_c = 4.20423\ldots$ for the Rössler system.

Now we estimate the scaling exponents for the Rössler oscillations analogously to the logistic map, but after discarding 10^6 transients and taking $N^* = 2^n$ Poincaré points on $x_{\max}$. The critical RQA measures are found as $RR_c = 0.2056$, $\langle L\rangle_c = 24.7783$ and $DIV_c = 0.0001$ for increasing N values at c_c (figs. 3(a), (b), (c), respectively). The power law scaling exponents are also similar to the theoretical value $\nu \approx 0.45$ and were estimated as $|RR - RR_c| \propto (c - c_c)^\alpha$ with $\alpha = 0.45 \pm 0.002$, $|\langle L\rangle - \langle L\rangle_c| \propto (c - c_c)^\beta$ with $\beta = 0.40 \pm 0.02$, and $|DIV - DIV_c| \propto (c - c_c)^\kappa$ with $\kappa = 0.49 \pm 0.001$ (fig. 3).

It is expected that the RQA scaling exponents (α, β and κ) for the logistic map and Poincaré section of the Rössler system should be equal in the range of their tolerance since these systems belong to the same universality class.

The value of neighbourhood threshold ε in the recurrence matrix $R_{i,j}(\varepsilon)$ affects the numerically obtained precision of the exponents. Therefore, showing these scaling relations and equality of RQA scaling exponents with the Huberman-Rudnick exponent ν is also important to define the most appropriate threshold value in RP.

The error described by the standard deviation of the slope ($\pm\sigma$) is decreasing, while the length of the time series is increasing (exemplarily shown for RR in fig. 4). This is related to determine the critical value of the measure, and so to determine the complete shape of the system at the chaos threshold, and the accuracy increases with increasing N.

Conclusions. – In summary, we have obtained scaling relations for the selected RQA measures that are similar to the universal Huberman-Rudnick scaling law $\nu \approx 0.45$ [30,32]. Although RQA measures have been frequently applied to study phase transitions (regime transitions), there is still a lack of a theoretical justification. Our analysis for the first time has filled this gap by considering their scaling behavior and critical exponents when approaching a critical transition point.

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