Although chaotic systems continue to gain interest, their confirmation and analysis can be difficult. Traditional analytic methods impose constraints which are often difficult to achieve. A technique which does not impose these constraints is recurrence quantification analysis. Recurrence quantification is derived from recurrence plots, which are based upon distances matrices of embedded series. The original article demonstrated the plot’s ability to uncover deterministic processes, as well as drift and nonstationarity. Recurrence quantification has allowed for direct quantification of these features.

Keywords: Recurrence quantification analysis; history.

1. Introduction

The last decade has witnessed curious developments in the analysis of dynamical signals: the original hope that chaos theory would help elucidate complex systems has met with some uncertainties. Initially, it was hoped that chaotic invariants could capture subtle nonlinear aspects of dynamical systems. But as more investigators become aware of the mathematical requisites (and limitations) of chaotic measures such as Lyapunov exponents and dimensions, they have recognized that new tools are needed. An important recognition in this respect is that many natural signals, in addition to being nonlinear, tend to be nonstationary, noisy and high dimensional. Certainly such a statement is not revolutionary, however, during a time when new, exciting concepts are emerging, it sometimes becomes easy to overlook basic facts, and to ignore fundamental assumptions [Zbilut et al., 2002].

In this context, a rather short, simple paper by Eckmann, Kamphorst and Ruelle was published [1987]. In evaluating a physical experiment, the authors embedded the time series in a higher dimensional space, and then plotted the recurrences in a distance matrix according to a rule defining an error tolerance. To their surprise, patterns were viewed which were previously not apparent in the original series (Fig. 1). What is remarkable about this method is that the algorithm requires no mathematical transformations or assumptions. Indeed, one of the purported uses for this method was to identify nonstationarities or changes of state.

Although the visual features of such plots are appealing, a drawback is their qualitative nature.
As a result, we set out to see if some of these features could be meaningfully quantified. Thus, we developed what has now been termed recurrence quantification analysis (RQA).

It is frequently remarked that recurrence quantification analysis is a rather new method of time series analysis. In some respects, such a statement is misleading. Although the conjunction of features employed by RQA is rather new, the basis for much of the mathematics has a long-standing history going back to Grassmann and metric spaces and geometric algebra in general [Fearnley-Sander, 1979]. An important underpinning is distance geometry with various embeddings and its combination with newer approaches in nonlinear dynamics. Also, the import of various techniques in time series analysis cannot be ignored.

2. Historical Context

The impetus itself can be traced at least to the late 60s and early 70s as scientists began to appreciate the time dependence of their various experimental measures. In this respect, the Fourier transform became an important technique, especially after the development of the Fast Fourier Transform (FFT) which greatly reduced the number of operations necessary for the computations [Cooley & Tukey, 1965]. The hope was that the FFT would provide answers to the understanding of data which had been largely ignored simply because of the massive amounts of data as well as calculations required. Even without the benefit of fast computers, the FFT provided a chance to perform some preliminary calculations. Certainly, the explosion of progress in computers in the 70s and 80s greatly benefited the area as FFT programs became increasingly accessible to more scientists.

Concomitantly, physicists started to become aware of the importance of nonlinear dynamics in describing a variety of time-dependent phenomena. Certainly, nonlinear dynamics was not new, but again, the development of high computational power allowed for a rapid development of understanding of these dynamics which were often only appreciated by numerical methods. Indeed, the classic story of Lorenz’s discovery of “chaos” is dependent upon findings for a set of coupled equations being iterated on a computer.

Unfortunately, as is common, the scientists in these various fields were not communicating with each other. Consequently, differences in terminology and customary practice often prevented adequate appreciation of discoveries. A classic example is the use of singular value decomposition which had been used for a long time by social scientists under the name of principle components or factor analysis.

Perhaps a common point of historical discussion is the FFT. Physical as well as biological scientists had begun to accelerate its usage in the 70s. Biologists especially were keen on its use due to the nature of the data they were inspecting. Time series plots of various physiological phenomena suggested some type of periodicity. Unlike physical data, physiological data tended not to show clear tendencies, save for obvious effects such as trends. It was hoped, therefore, that the FFT could unravel some of the involved frequencies. In this respect, heart rate variability became a focus of interest.

Heart rate variability became an issue perhaps for three reasons: (1) there was a theoretical basis for the suggestion of embedded periodicities; (2) large samples of heart rate were relatively easy to collect; and (3) there was a hope that its analysis could provide a basis for evaluating pathology. An important landmark in this effort was the publication of articles dealing with heart rate variability [Sayers, 1973; Lueck & Lurig, 1973]. However, it was not long before the initial optimism began to diminish. Although the FFT can provide...
instrumentally sharp peaks for physical processes, the nature of biological (and many other natural) processes is that they are inherently noisy, or at least derivative of multiple processes. Furthermore, the FFT requires at least relative stationarity which can be difficult to obtain in living organisms which are constantly adapting to their environment. An additional difficulty was the recognition that many biological processes are nonlinear, and cannot be resolved by linear transforms such as the FFT.

Quite naturally then, as developments grew in the area of nonlinear science and chaos, more and more attention was paid to the possibility that many of these phenomena might be examples of such systems. Clearly, these systems were deterministic, yet not in the traditional sense. Adding to the speculation was the fact that many of the proffered time series appeared very similar to many natural processes. The difficulty, of course, was demonstrating the possibility of chaotic dynamics in natural processes. Among the early disseminators of nonlinear dynamics into the life sciences were Mandell [Koslow, 1973] and Goldberger et al. [1986].

Certainly, a milestone was the publication of Grassberger and Procaccia’s correlation dimension [Grassberger et al., 1991] which seemed to present the possibility of providing objective proof of the (non)existence of chaos for a given dynamical system. The late eighties and early nineties witnessed an explosion of efforts calculating the so-called “G-P” dimension. However, the difficulties which plagued the FFT soon caught up with the correlation dimension, as well as the related calculation of Lyapunov exponents. Problems of size, stationarity, etc., attended these techniques as well [Ruelle & Eckmann, 1992].

Our own concerns developed when we tried to calculate dimensions for heart rate variability even under circumstances which were felt to be ideal. Even when stationarity and relatively long data sets were obtained, dimension calculations managed to obtain relatively large error bars so as to preclude chaotic certitude (Fig. 2) [Zbilut & Mayer-Kress, 1988].

At approximately the same time, Eckmann, Kamphorst and Ruelle published their relatively short paper — almost as an afterthought to their efforts in establishing a method for Lyapunov calculation. They presented a graphical method for displaying “recurrences” (essentially the basis for correlation dimension and Lyapunov exponent calculation). Their intent was to demonstrate the ability to obtain useful information for understanding the dynamics of a studied system. Thus they suggested that such plots were important not only for the demonstration of determinism, but also stationarity and drift. They also pointed out that the largest positive Lyapunov exponent was related to the length of observed line segments. Moreover, the graph itself is really a time series distance matrix based upon an embedding metric. This is to say that each ordered index point in the graph represents the time series itself (see Fig. 3). The basic relation for a dynamical system represented by a trajectory \(\{\vec{x}_i\}\) for \(i = 1, \ldots, N\) in a \(d\)-dimensional phase space is:

\[
R_{i,j} = \Theta(\varepsilon - |\vec{x}_i - \vec{x}_j|), \quad i, j = 1, \ldots, N, \tag{1}
\]

where \(\varepsilon\) is a defined threshold and \(\Theta(\cdot)\) is the Heaviside function. From this standpoint, recurrence plots are not really new — rather they are a fortuitous combination of different mathematical traditions. The one drawback they had was the fact that they were still qualitative. Especially, when factors such as screen and printer resolution were taken into consideration, a certain amount of ambiguity persisted — especially when the human penchant for seeing patterns everywhere is taken into consideration. The features identified by Eckmann et al. begged for some objectivity.

This was our motivation in the late eighties when we began an attempt to quantify these features [Zbilut et al., 1991; Zbilut & Webber, 1992; Webber & Zbilut, 1994]. Over the span of some five years, we ended up quantifying many of these
features, as well as demonstrating their usefulness especially for real, experimental data (Fig. 3). Our efforts were spurred with the publication of a brief article by Ruelle in which he opined that pursuing demonstrations of “chaos,” without any practical reason to do so may be irrelevant [Ruelle, 1994]. In other words, nonlinear dynamics can be useful to understand, but to specifically “demonstrate” chaos without any clear purpose becomes a fatuous endeavor. The implication is that saying something is chaotic is similar to saying something is normally distributed — without further measurements, analysis, etc., the relevance is lacking. By avoiding the problem of having to say something is or is not chaotic and going directly to measurement as with RQA, one can immediately determine if there is anything interesting in the dynamics.

During this time there have been improvements, as well as other algorithms seeking to quantify recurrences. They all appear to recognize the importance of recurrences as the basis for any meaningful understanding of any time-varying system [Gao & Cai, 2000; Choi et al., 1999; Marwan et al., 2002].

3. Conclusions

At this point, we are gratified by the growing interest in recurrence analysis (Fig. 4). Indeed, even a cursory examination of publications using these techniques demonstrates a wide variety of scientific disciplines, from astronomy to molecular biology.

Fig. 4. Growth of use of recurrence analysis.
Indeed, even more quantifications and analyses have been developed.¹

An important point is that although there is the temptation to over-state the advantages of RQA, in many circumstances, methods such as the FFT can be very useful when there is a large, clear effect such as in the case of approximate periodicities. However, when there are nonstationarities and/or nonlinearities, such as are often encountered in real data, which result in important, but perhaps subtle effects, methods such as RQA should be considered.

References


¹For an excellent updated bibliography maintained by N. Marwan, see http://www.recurrence-plot.tk.