Research paper

Recurrence quantity analysis based on matrix eigenvalues

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Abstract

Recurrence plots is a powerful tool for visualization and analysis of dynamical systems. Recurrence quantification analysis (RQA), based on point density and diagonal and vertical line structures in the recurrence plots, is considered to be alternative measures to quantify the complexity of dynamical systems. In this paper, we present a new measure based on recurrence matrix to quantify the dynamical properties of a given system. Matrix eigenvalues can reflect the basic characteristics of the complex systems, so we show the properties of the system by exploring the eigenvalues of the recurrence matrix. Considering that Shannon entropy has been defined as a complexity measure, we propose the definition of entropy of matrix eigenvalues (EOME) as a new RQA measure. We confirm that EOME can be used as a metric to quantify the behavior changes of the system. As a given dynamical system changes from a non-chaotic to a chaotic regime, the EOME will increase as well. The bigger EOME values imply higher complexity and lower predictability. We also study the effect of some factors on EOME, including data length, recurrence threshold, the embedding dimension, and additional noise. Finally, we demonstrate an application in physiology. The advantage of this measure lies in a high sensitivity and simple computation.

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1. Introduction

For a large number of scientific disciplines, such as astrophysics, biology or geosciences, we can do some data analysis to understand the complex process observed in nature. Different types of systems, from very large to very small time scales can be modeled by differential equations. In principle, we can predict the state of such a system with arbitrary precision once the initial conditions are known. It is because these systems always evolve in a similar way or occur over and over again [1,2]. However, some complex systems are very sensitive to fluctuations and even the smallest perturbations of the initial conditions can make a precise prediction on long time scales impossible. Linear approaches of time series analysis are often insufficient, and most nonlinear techniques [3–8], such as fractal dimensions [9] or Lyapunov exponents [10,11], suffer from the curse of dimensionality and need long data series. Therefore, the application of these methods, especially for short time series, can lead to serious pitfalls. Entropy, to a certain extent, can show the complexity of a system, such as the Rényi entropy [12], the renormalized entropy [13] and so on.

In this paper, we focus on another technique of complexity measure, which is based on the method of recurrence plots (RPs). In 1987, the method of RPs was first introduced by Eckmann et al. to visualize the recurrences of dynamical systems, which can be portrayed by a trajectory \( \{ x(i) \}_{i=1}^{N} \) in its phase space [14]. Then, the corresponding RP is defined as the following

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matrix:

\[
R(i, j) = \begin{cases} 
1 & \text{if } x_i \approx x_j \\
0 & \text{if } x_i \neq x_j
\end{cases} \quad i, j = 1, 2, \ldots, N
\]

(1)

where \( x_i \approx x_j \) means equality up to a cut-off distance \( \epsilon \). That is to say, the matrix reflects whether the states of a system at times \( i \) and \( j \) are similar or not. If the states are similar, \( R(i, j) = 1 \). On the contrary, if the states are rather different, the corresponding entry in the matrix \( R(i, j) = 0 \). So this matrix can tell us when the system’s state will appear again. This approach has been used to analyze non-stationary and short data series [15]. RPs have been considered as a powerful technique to reveal the statistical properties of the system through the structural features of RPs. Cross recurrence plot (CRP) [16,17] and joint recurrence plot (JRP) [18,19] were proposed as bivariate extension of the RP. CRPs reveal valuable information about the relationship between both systems by comparing their states represented in two time series. JRP compare different systems by considering the recurrences of their trajectories in their phase spaces separately and look for the times when both of them recur simultaneously.

Beyond the visual impression yielded by RPs, some measures for recurrence plots have been proposed and used to detect typical transitions occurring in complex systems, which are known as recurrence quantification analysis (RQA). RQA reflects the nonlinear properties over the recurrence point density, the diagonal and the vertical line of the RPs [20–26]. RQA contains several measures: measures based on the recurrence density–recurrence rate (RR), measures based on diagonal lines—determinism (DET), the average diagonal line length (L), divergence (DIV), entropy (ENTR) and so on; measures based on vertical lines—laminarity (LAM), trapping time (TT) and the maximal length of the vertical lines (Vmax). Compared to many classical nonlinear analysis methods, RQA overcomes some limitations, which does not require large data sizes and is less affected by noise and non-stationarity [20,27].

RQA based on structural characteristics of RP is very mature. Recurrence quantity analysis based on singular value decomposition is studied in [28]. Considering RP is a symmetric and binary matrix and matrix eigenvalues carry a lot of basic information about the complex systems, we study the RP from the aspect of matrix eigenvalue. There are many eigenvalues of RP close to zero, and the greater the proportion of eigenvalues approaching zero is, the higher the stability and predictability of the system are. For example, the ratio of the eigenvalues close to zero for periodic system is bigger than that of the non-periodic system. Because the eigenvalues close to zero are quite different in the order of magnitude, we put all the eigenvalues of the logarithmic scale. In general, the eigenvalues can be divided into three parts by a logarithmic scale. Our idea is to reflect the characteristics of the system through the complexity of eigenvalues of RP. The Shannon entropy is a measure to assess the complexity of a dynamical process and can be used to quantify transitions between different dynamical regimes, so we put forward the entropy of eigenvalues (EOME) for RPs based on Shannon entropy.

We verify that the measure EOME can quantitatively describe the RPs through experiments, and it is especially helpful to find various transitions in dynamical systems. For periodic system the value of EOME is rather small, indicating its low complexity and high predictability. However, with an increasing chaotic nature of the system the EOME values will increase. We find that EOME is sensitive to noise, so it is necessary to ensure that the data is not disturbed by noise. The influence of other factors on EOME is also studied. Logistic map, as a typical example of complex systems, is used by us to verify that EOME can measure the characteristics of different systems. Compared with other RQA metrics, EOME is a better choice to detect the transitions from periodic to chaotic and chaotic to periodic states.

This paper is organized as follows. In Section 2, the definitions of RP and EOME are proposed. In addition, we introduce some features of RPs and how to calculate EOME of RP. Section 3 is devoted to prove EOME can be used to distinguish different systems. The effects of some factors on EOME are also studied in this section. In Section 4, the logistic map is used to demonstrate that EOME can detect the transitions between different systems. Furthermore, we demonstrate an application of EOME in physiology. We summarize and give our conclusions in Section 5.

2. Methodology

2.1. Recurrence plots

In our daily life, some situations occur over and over again. Similarly, in some systems, some conditions occur over and over again. Recurrence plot is a tool which measures recurrences of a state. Recall that we have a time series \( \{u_t\}_{t=1}^N \) with the length \( N \). After choosing the time delay \( \tau \) and embedding dimension \( m \), we can express the dynamics with a reconstruction of the phase space trajectory \( \tilde{x}_t \) from a time series \( \{u_t\}_{t=1}^N \) [29,30]:

\[
\tilde{x}_t = (u_t, u_{t+\tau}, \ldots, u_{t+(m-1)\tau}) \quad t = 1, 2, \ldots, N - (m-1)\tau
\]

(2)

Two embedded parameters, the dimension \( m \) and delay \( \tau \), must be chosen appropriately. Methods for the estimation of the smallest sufficient embedding dimension (e.g. false nearest neighbours [31]) and for an appropriate time delay (e.g. the auto-correlation function, the mutual information function [32,33]) have been proposed.

In phase space, for a given trajectory \( \tilde{x}_i \), the recurrence plots are defined as [14,34]:

\[
R_{i,j}(\epsilon) = \Theta(\epsilon - ||\tilde{x}_i - \tilde{x}_j||), \quad i, j = 1, 2, \ldots, N
\]

(3)

where \( \tilde{x}_i \) is a state point of a system in a phase space, \( \epsilon \) is a threshold distance, \( \Theta(\cdot) \) is the Heaviside function and \( ||\cdot|| \) is a norm. The most frequently used norms are the \( L_1 \)-norm, the \( L_2 \)-norm (Euclidean norm) and the \( L_\infty \)-norm (Maximum or
In this paper, we use the Euclidean norm. RPs is a $N \times N$ grid of points, which are encoded as black for 1 and white for 0. RPs is a symmetric and binary matrix, in which the values of the main diagonal are all one ($R_{i,i}(\varepsilon) = 1$).

From the Eq. (2), we know that the RP is affected by the threshold $\varepsilon$. If the threshold $\varepsilon$ is too small, then there will be almost no recurrence points, so that we can not get the anything about the system from the RP. On the other hand, if the threshold $\varepsilon$ is too large, each point can be a recurrence point and the RP is filled with black dots. So we have to choose a suitable threshold $\varepsilon$. How to choose a suitable threshold value has been studied in many literatures [21,34-38].

Different systems have different recurrence plots, so we can judge the characteristics of the system by the structure of the recurrence plots [34]. Single points appear in the RP, which shows the state does not persist for a long time, that is, the system has a high volatility. Usually a RP consisting of only discrete points is related to white noise. Diagonal lines of length $L$ defined by $R_{i,k,j+k}(\varepsilon) = 1$ (for $k = 1...L$), represent that the trajectory visits the same region of phase space at distinct times. Vertical and horizontal lines [39] of length $L$, expressed by $R_{i,j+k}(\varepsilon) = 1$ or $R_{i+k,j}(\varepsilon) = 1$(for $k = 1...L$), display that the state does not change or changes slowly in time, namely, the state of the system is trapped for some time by this texture. Many works have studied and summarized the relationship between the structure characteristics and the system, and also proposed some measures to quantify the structural characteristics [40].

2.2. Recurrence quantification analysis based on matrix eigenvalue

As long as the RP is determined, the corresponding eigenvalues are determined. Here we assume a RP of $n \times n$ with the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. In the study, we find that some of the eigenvalues of $R_{i,j}$ are almost zero, and the absolute values of the other eigenvalues are relatively large with respect to zero. While some of the eigenvalues are almost zero, they vary greatly in magnitude, like $10^{-30}$ and $10^{-40}$. In order to show the differences between these eigenvalues more clearly, we deal with these eigenvalues by:

$$\sigma_i = \ln(|\lambda_i|)$$  \hspace{1cm} (4)

Log-eigenvalues spectrum (see Fig. 1(a)), which describes the distribution of $\sigma_i$, separates the eigenvalues into three parts (the left part, the middle part and the right part). Sometimes, the log-eigenvalues spectrum may be only two parts (no the left part). The greater the proportion of the middle part, the better the stability of the system is, and the higher the predictability of the system is. We introduce a new metric to describe the relationship between eigenvalues and systems. Considering entropy quantifies the disorder of the system, therefore, we put forward the entropy of matrix eigenvalues (EOME) based on eigenvalues.

Firstly, we obtain the eigenvalues $[\lambda_i]_{i=1}^{N}$ arranged in the order from small to large and get the log-eigenvalues of the recurrence matrix. Then, according to the log-eigenvalue spectrum (see Fig. 1), we divide the original eigenvalue into three parts (the left part $\lambda^1$, the middle part $\lambda^m$ and the right part $\lambda^r$).

We calculate the proportion of the three parts according to the log-eigenvalues spectrum.

$$POL = \frac{N_{\lambda^1}}{N};$$  \hspace{1cm} (5)
\[
POM = \frac{N_{s \text{m}}}{N}; \tag{6}
\]
\[
POR = \frac{N_{s \text{r}}}{N}; \tag{7}
\]
Here, \(N_{s \text{m}}, N_{s \text{m}} \) and \(N_{s \text{r}} \) are the length of the sequence \(\lambda^1, \lambda^m, \lambda^l \), respectively. After that, we calculate the EOME of each part. The measure EOME of each part refers to the Shannon entropy of the probability of the eigenvalues belongs to the three parts,
\[
EOME_{\text{left}} = - \sum_{i=1}^{n_1} (p^l(i) \text{POL}) \ln(p^l(i) \text{POL}) \tag{8}
\]
\[
EOME_{\text{middle}} = - \sum_{i=1}^{n_2} (p^m(i) \text{POL}) \ln(p^m(i) \text{POL}) \tag{9}
\]
\[
EOME_{\text{right}} = - \sum_{i=1}^{n_3} (p^r(i) \text{POR}) \ln(p^r(i) \text{POR}) \tag{10}
\]
Here, \(p^l, p^m \) and \(p^r \) are the probability distribution of the eigenvalues of the left part, the middle part and the right part, respectively. Since all the eigenvalues of the middle part are very close to zero, we can think that \(n_2 = 1 \) and \(p^m(0) = 1 \). Therefore, the modified \(EOME_{\text{middle}} \) is:
\[
EOME_{\text{middle}} = - \text{POM} \ln(\text{POM}) \tag{11}
\]
To obtain the probability distribution of the left part and right part, we first carry on the symbolic processing to the original eigenvalues sequence, then study the statistical characteristics of the symbolic sequence. The parameter \(n_1 \) (\(j=1,3 \)) is the number of different symbols for each symbol sequences. Here, the max-min-method \([41]\) is used to obtain symbol sequences. For example, for the sequence \(\lambda^1 \), we obtain its symbolic sequence \(S_{\lambda^1}^1 \) that comprises the full range of dynamics of the original eigenvalues sequence. The difference between the minimum and the maximum of the sequence \(\lambda^1 \) is divided into a \(n_1 \) quantization bins. According to Scott’s choice \([42,43]\), we set \(l = 3.5\sigma_{\lambda^1}/N_{\lambda^1}^{1/3} \) as the size of each bin, where \(\sigma_{\lambda^1} \) is the standard deviation of the sequence \(\lambda^1 \). So, the number of the bins is: \(n_1 = (\max(\lambda^1) - \min(\lambda^1))/l \). Hence, the transformation is as follows:
\[
S_{\lambda^1}(i) = \begin{cases} 
0 & \min(\lambda^1) \leq \lambda^1(i) < \min(\lambda^1) + 1 \cdot l \\
1 & \min(\lambda^1) + 1 \cdot l \leq \lambda^1(i) < \min(\lambda^1) + 2 \cdot l \\
\vdots & \vdots \\
n_1 - 1 & \min(\lambda^1) + (n_1 - 1) \cdot l \leq \lambda^1(i) \leq \max(\lambda^1) 
\end{cases} \tag{12}
\]
After getting the symbol sequence \(S_{\lambda^1} \), we estimate the probability distribution \(p^l \) of the original sequence \(\lambda^1 \) by calculating the frequency of each symbol in the symbol sequence. Similarly, we can also get the \(p^m \) and \(p^r \).

Then, the EOME of the whole system is:
\[
EOME = EOME_{\text{left}} + EOME_{\text{middle}} + EOME_{\text{right}} \tag{13}
\]
Why do we divide the eigenvalues into three parts and then calculate EOME, instead of calculating EOME directly by those eigenvalues? These eigenvalues vary greatly in magnitude, so it will cause great errors if we directly calculate the EOME. The eigenvalues are divided into three parts, which not only takes into account the differences of individuals but also can accurately reflect the dynamic characteristics of the system.

As a given dynamical system changes from a non-chaotic to a chaotic regime, it is expected that the entropy increases as well, since it is a measure of complexity. In our study, we find that the higher the complexity of the system is, the greater the corresponding EOME is, and vice versa. So, EOME reflects the complexity of the RP, e.g. \((1) \) for periodic system the value of EOME is rather small, indicating its low complexity; \((2) \) with increasing chaotic nature of the system, the EOME values will increase. Moreover, the EOME measure is especially helpful to find various transitions in dynamical systems.

We all know that RQA measurements strongly depend on the embedding dimension \([35,44]\). The embedding dimension is increased, and the recursive point is reduced, which can affect the structure characteristics of the RP. On the other hand, data size also affects the structure of RP. For example, considering a periodic dynamic, the RP is composed of infinitely long diagonal lines in principle. However, the diagonal on the boundary of RP will be cut off due to the finite time series length. This biasing effect can also happen for non-periodic dynamics as long as a significant number of diagonals cross the border of the RP. EOME measure is also affected by the embedding dimension and the data size. The content and size of the recurrence matrix can be changed with the change of the embedding dimension and the length of the data, which can cause the change of eigenvalues.
Fig. 2. Characteristic typology of recurrence plots: (a) homogeneous (uniformly distributed white noise), (b) periodic (a sine function: \( \sin(2\pi t) \)), (c) drift (logistic map corrupted with a linearly increasing term \( x_{t+1} = 4x_t(1 - x_r + 0.01t) \)) and (d) disrupted (Brownian motion). These examples illustrate how different RPs can be. The used data have the length 500 (a, b, d) and 200 (c), respectively; RP parameters are \( m = 1, \tau = 1, \varepsilon = 0.1\sigma; L2\text{-norm} \).

Fig. 3. Eigenvalue spectrum for these four characteristic typologies. The shape of the eigenvalue spectrum for different systems is roughly similar. So, it is difficult for us to judge the characteristics of the system by the eigenvalue spectrum.

3. Simulation experiment

3.1. Four characteristic typologies of RP

We know that typology and textures are two important characters of RP. Here, we focus on the typology, which is the large scale patterns in RPs. It can be classified in homogeneous, periodic, drift and disrupted ones [34,45]:

1. Homogeneous, RPs are typical of stationary and autonomous systems. An example of such an RP is that of a random time series (Fig. 2(a)).
2. Periodic (see Fig. 2(b)), characterized by diagonal lines, which have the same periodic distance from each other. These are typical of periodic systems.
3. Drifts (see Fig. 2(c)), caused by systems with slowly varying parameters. Such slow change makes RP pale in the upper-left and lower-right.
4. Disrupted, white areas or bands (see Fig. 2(d)), indicating non-stationarity, abrupt changes as well as extreme events in the dynamics.

Different structural features of RPs are displayed in Fig. 2 for the four characteristic typologies. Then we obtain the eigenvalues of each RPs shown in Fig. 2. Here, the length of each sequence is 500, and Fig. 3 describes the eigenvalue spectrum for each RPs. We find that the eigenvalue spectrum is similar and most of the eigenvalues are near zero. To better distinguish the four different types of data, the result for the log-eigenvalues is obtained and is shown in Fig. 4. Through observation, the log-eigenvalues spectrum is basically composed of three parts, and the greater the proportion of the middle part (the braces lying) is, the lower the complexity of the system is, which indicates a high predictability of the system. Then we calculate the EOME and some RQA measures corresponding to each RP, and the results are shown in Table 1. It’s easy to find that the EOME of the periodic system is obviously smaller than that of the other systems, implying the periodic system has a high predictability and low complexity.
However, same is known of eigenvalues that has results (3.2.1. (old, 3.2. Fig. 20 of (the eigenvalues t With The amount of the amount of data leads to the change of the eigenvalue distribution, which makes the EOME change.

### 3.2. Effects of some factors on EOME

The purpose of this part is to evaluate the influence of some factors on EOME, including data length, recurrence threshold, the embedding dimension, and additional noise. We use Gauss white noise and two simple periodic oscillation models \( f(t) = \sin(2\pi t) \) (\( T = 1 \)), \( f(t) = \sin(0.5\pi t) \) (\( T = 4 \)) to study how EOME depends on these factors.

#### 3.2.1. Dependence of EOME on the data length

With the embedding dimension \( m = 1 \), the time delay \( \tau = 1 \), the recurrence threshold \( \varepsilon = 0.1\sigma \), and the experimental results about the influence of the amount of data on EOME are shown in Fig. 5(a). The effect of data length on EOME is small for WGN with a stable value for different data size. It can be found that, for the periodic system (see Fig. 5(a)), EOME has a downward trend with the increase of the amount of data. This is mainly because the proportion of the eigenvalues that belong to the middle part also increases (Fig. 5(b)) with the increase of the amount of data. We know that the sum of eigenvalues of a recursive matrix is equal to the number of eigenvalues. The large POM implies that only a small number of eigenvalues are nonzero. Therefore, there will be extreme cases, that is, some of the eigenvalues are very large. It is well known that the Shannon entropy has a maximum value when the data obeys uniform distribution. The larger the POM is, the more uneven the distribution of the eigenvalues is. So EOME will decrease with the increase of data length. At the same time, we can get some additional results from Fig. 5(a). The value of EOME of long period systems is relatively large. However, it is found that for the system with large period, we need more data to estimate the EOME. In fact, when the data

![Fig. 4. Log-eigenvalue spectrum for these four characteristic typologies. The Log-eigenvalue spectrum is composed of three parts, but the proportion of each part is not the same for different systems.](image)

<table>
<thead>
<tr>
<th>Table 1</th>
<th>The EOME and some RQA measures for four characteristic typologies of RP.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Homogeneous</td>
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<tr>
<td>RR</td>
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</tr>
<tr>
<td>DTE</td>
<td>0.1068</td>
</tr>
<tr>
<td>ENTR</td>
<td>0.2484</td>
</tr>
<tr>
<td>TT</td>
<td>0</td>
</tr>
<tr>
<td>EOME</td>
<td>1.8202</td>
</tr>
</tbody>
</table>

![Fig. 5. (a) Influence of data length on the EOME. The amount of data has little effect on EOME for WGN. (b) The effect of data length on POM. The increase of the amount of data leads to the change of the eigenvalue distribution, which makes the EOME change.](image)
almost the embedding oscillating percentage

3.2.3. dimension analysis detect maximum direction rectly

3.2.2. is Fig.

Fig. 6. (a) RP for WGN, embedding dimension \( m = 4 \). Only a few single points and main diagonal appear in the RP. (b) RP for signal \( f(t) = \sin(2\pi t) \). embedding dimension \( m = 2 \). Compared with Fig. 2(b), only the 45° direction diagonal structure appears in RP. (c) Influence of embedding dimension on the EOME for WGN and periodic systems.

is 500, the EOME is not accurate and is too large for a system with \( T = 4 \). There is not enough data so that the structural features of the RP cannot be fully displayed.

3.2.2. Influence of embedding dimension on the EOME

The selection of different embedding dimension for time series will directly affect the structure of the RPs. For WGN, in the case of \( m = 4 \), \( \tau = 1 \), \( \varepsilon = 0.1\sigma \), the RP contains only a few single points and main diagonal (see Fig. 6(a)). In particular, the effect of embedding dimension on the RPs of oscillating signals is obvious. The RP contains diagonal lines in 45° direction but also –45° direction for non-embedding (as shown in Fig. 2(b), \( m = 1 \)), but only the 45° direction when correctly embedded (see Fig. 6(b), \( m = 2 \)). With the increase of the dimension, the recurrence points on the diagonal of –45° direction will slowly disappear. The effect of embedding dimension on EOME is shown in Fig. 6(c). EOMEs reach a positive maximum for \( m = 2 \), but when the embedding dimension is greater than 5, the EOMEs of WGN are 0. In the case of \( m \geq 5 \), all the recursive points appear on the main diagonal, which means that all the eigenvalues are 1. At this time, EOME fails to detect any potential information about the system. Therefore, we can not choose too large embedding dimension for EOME analysis for the systems with high randomness. For oscillating signals, we can find that EOME increases with embedding dimension only for small embedded value and tends to be stable when the dimension is greater than 4.

3.2.3. The influence of noise and recurrence threshold on EOME

Noise would distort any existing structure in the RP. How does noise affect EOME? We add WGN realizations to the two signals \( f(t) = \sin(2\pi t) (T = 1) \), \( f(t) = \sin(0.5\pi t) (T = 4) \). The standard deviation of the WGN realizations is assigned as a percentage of the original signal sequences (from 0 to 200%). Results, shown in Fig. 7(a), indicate that The EOME analysis of oscillating signals is very sensitive to noise. The values of the EOME for oscillation signals with additional noise are similar.
to that for WGN. Therefore, in the presence of noise, EOME analysis may have the wrong results. So we must filter out the noise from the original signal when obtaining the data.

The increase of the threshold $\varepsilon$ will increase the number of recurrence points in the RP. However, if $\varepsilon$ is chosen too large, almost every point is a neighbour of every other point, which leads to some false recurrence. Let the threshold $\varepsilon$ gradually increase from $0.1\sigma$ to $0.3\sigma$, we calculated the EOME for WGN and periodic signals. Fig. 7(b) shows the results of the experiment. For periodic signals, EOME has a slight downward trend with the increase of threshold. For a periodic system, an increase in the threshold will cause the point around the diagonal in RP to be recurrence points. However, there is little change in the distribution of eigenvalues. Therefore, EOME is not sensitive to the threshold for periodic signals. The change of EOME is a little bigger for WGN (see Fig. 7(b)).

4. Application

4.1. Logistic map

In this part, we use EOME to detect the dynamic changes of a complex system. It is well-known that logistic map is a useful tools for studying the chaotic behavior. With the continuous change of the parameter $a$, various regimes and transitions between them occur, e.g., accumulation points, periodic, and chaotic states, band merging points, period doublings, and inner and outer crises [39,46–48]. The logistic map is defined by

$$x_{i+1} = ax_i(1 - x_i)$$  \hspace{1cm} (14)

where $x_i$ is a real number and ranges from zero to one. Here, we analyze the logistic map within the interesting range of the control parameter $a \in [3.5, 4.0]$ with a step size of $\Delta a = 0.0005$. For $3.5 < a < 4$, the logistic map shows interesting behavior such as repeated period doubling, appearance of odd periods, and for $a = 4$ the logistic map is chaotic. For each $a$ we compute a time series of the length $N=2000$. In order to exclude transient responses, we use the last 1000 values of these data series in the following experiment. Here, we use an embedding of $m = 3$ and $\tau = 1$. The cut-off distance $\varepsilon$ is selected to be 10% of the standard deviation of the series.

For various values of the control parameter $a$ we will get RPs with different features. Here we select four different parameter values ($a = 3.679, 3.720, 3.830$ and $4.000$) to show the different features of the RPs. The original data is displayed in Fig. 8, and the RPs in these states are shown in Fig. 9. For the periodic states ($a = 3.830$), as shown in Fig. 9(c), continuous and periodic diagonal lines appear in the RP, but no vertical or horizontal lines. On the other hand, chaos-chaos transitions ($a = 3.679$ and $3.720$), band merging points inner crises and regions of intermittency represent states with short laminar behavior and cause vertically and horizontally spread black areas in the RP (Fig. 9(a) and (b)). The fully developed chaotic state ($a = 4$) causes a rather homogeneous RP with numerous single points and some short diagonal or vertical lines (Fig. 9(d)). RQA measures, based on the recurrence point density and the diagonal and the vertical line of the RP, can distinguish different dynamical behavior of the systems, such as chaos-period and chaos-chaos transitions. Next, we use the measures we proposed to test whether they are able to effectively distinguish these states. Now, we have a test on EOME measure. Firstly, we compute the eigenvalues of the recursive matrix in these four different states, and deal with these eigenvalues in a logarithmic scale. Fig. 10 shows the log-eigenvalue spectrum. The span of braces lying, the middle part of the log-eigenvalue spectrum, is largest for the third states (see Fig. 10(c)), and is smaller for other states, which indicates that the system is stable in third state and is chaotic or transformed into chaos for other states. We compare the EOME for these four different
Fig. 9. Recurrence plots for the system in these four different states: band merging $a = 3.679$ (a) and $a = 3.72$ (b); period window $a = 3.83$ (c); and chaos $a = 4$ (d), with an embedding of $m=3$ and $\tau = 1$, and $\varepsilon = 0.1\sigma$. RPs show different structures in different states. To better illustrate the diagonal structure of the (c), we select the data length as 250.

Table 2

The EOME and some RQA measures for logistic map with different parameter values ($a = 3.679$, 3.720, 3.830 and 4.000).

<table>
<thead>
<tr>
<th></th>
<th>$a = 3.679$</th>
<th>$a = 3.720$</th>
<th>$a = 3.830$</th>
<th>$a = 4.000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR</td>
<td>0.0538</td>
<td>0.0476</td>
<td>0.3333</td>
<td>0.0351</td>
</tr>
<tr>
<td>DTE</td>
<td>0.8130</td>
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<td>ENTR</td>
<td>1.9782</td>
<td>1.9596</td>
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<tr>
<td>TT</td>
<td>4.7516</td>
<td>3.6400</td>
<td>0</td>
<td>2.5132</td>
</tr>
<tr>
<td>EOME</td>
<td>1.7014</td>
<td>1.8646</td>
<td>0.0563</td>
<td>1.8943</td>
</tr>
</tbody>
</table>

states and the results are shown in Table 2. Table 2 also displays a comparison on EOME and several typical RQA measures. The main finding is that EOME gradually increases from the periodic state to chaotic state. So the measure of EOME can detect the transitions from periodic to chaotic and chaotic to periodic states.

Then we explore the change of EOME as the parameter increased from 3.5 to 4 with a step size of $\Delta a = 0.0005$. The values of EOME within periodic windows are consistent, e.g., for $a \in [3.50, 3.544]$, $a \in [3.544, 3.567]$ and $a \in [3.82, 3.845]$, and smaller than during chaotic regimes (see Fig. 11(f)). At the critical values of $a = 3.567$, 3.633, 3.743, and 3.8455, the values of EOME reveal sharp jumps. Furthermore, this measure shows a rapid decrease and increase at the transitions between chaos and periodic windows, and maxima or peaks at the chaos-chaos transitions. Comparing to the RQA measures, we can find that EOME is more effective in distinguishing between the periodic state and chaotic state.
4.2. EOME analysis in ECG signals

As is known to all, RPs and the RQA are most popular in physiology [20]. With the continuous improvement of RPs and the RQA, various successful applications in other fields of life science, as neuroscience and genomics [39,49–53], have also been published. Of course, they have also been successfully applied to other areas, like biology [54,55], physics, ecology, earth science [16,44,56], chemistry and astrophysics [57,58], engineering [59,60] and economy. In this section, we apply EOME to physiological data. The goal is to illustrate the capabilities of EOME to detect transitions in measured physiological data.

We get the ECG (electrocardiogram) signals from: http://www.physionet.org/cgi-bin/atm/ATM. ECG is the most commonly used clinical detection to diagnose myocardial ischemia and myocardial infarction. This section, we download the ECG data with the length 2000 for 20 people (ten healthy samples and ten healthy samples). Each record includes 15 simultaneously measured signals: the conventional 12 leads (i, ii, iii, avr, avl, avf, v1, v2, v3, v4, v5, v6) together with the 3 Frank lead ECGs (vx, vy, vz). We carry out the RQA and EOME analysis on those ECG data. The results find that the mean of EOMes for ECG signals from v4, v5, and v6 can well reflect differences between patients and normal individuals. Here we show the results of the two ECG signals from a healthy person and an unhealthy person.

With an embedding of m=3 and τ = 1. and $\varepsilon = 0.1\sigma$. Fig. 12 shows the RPs of the ECG signal from healthy and unhealthy person. Both the diagonal structure and the vertical line structure appear in the RPs. However, RPs of the ECG signal differ from each other for healthy person and unhealthy person. Block structure is uniformly distributed in the RP of healthy person, and the line structure in Fig. 12(a) is much more than that in Fig. 12(b), showing that the ECG signal of a healthy person is more stable. By observing the log-eigenvalue spectrum, the proportion of the middle part of healthy person in
Fig. 11. EOME and some selected RQA measures (RR, DET, ENTR, LAM and TT) for logistic map. We can find that these measures are constant in some periods of the window, such as $a \in [3.50, 3.544]$ and $a \in [3.82, 3.84]$. With the change of the parameter $a$, these measures also show complex fluctuations.

Fig. 13(a) is more than that of unhealthy person (see Fig. 13(b)), which implies that the complexity of the ECG signal of healthy person is lower.

Some RQA measures and EOME are computed, and the results are shown in Fig 14. For RQA’s experimental results (see Fig. 14(a), (b) and (c)), it can be found that the indicators (ENTR, LAM, TT) of healthy individuals are larger than those of diseased individuals, which shows that the RPs of healthy individuals have more diagonal and vertical line structures than unhealthy individuals. The distributions of the experimental results of RQA measures exist crossover, so these measures do not well distinguish diseased individuals from these samples. The EOMEs of healthy individuals are obviously smaller than that of diseased individuals, which indicates that the ECG signals of healthy person are more stable and have a low complexity. Therefore, EOME can effectively identify the ECG signal of healthy person and unhealthy person.
Fig. 12. The recurrence plots for the ECG signals from a healthy person (a) and an unhealthy person (b). It is easy to find that the black points are more evenly distributed in (a), which shows that the ECG signal for healthy people is more stable.

Fig. 13. Log-eigenvalue spectrum for the recurrence matrix of a healthy person (a) and an unhealthy person (b).

Table 3
The EOMEs for 10 unhealthy samples and 10 healthy samples. The rank of EOME is obtained from the order of all samples from small to large.

<table>
<thead>
<tr>
<th>Sample number</th>
<th>Unhealthy samples (rank)</th>
<th>Healthy samples (rank)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.8629 (17)</td>
<td>1.4985 (2)</td>
</tr>
<tr>
<td>2</td>
<td>1.8274 (15)</td>
<td>1.6115 (7)</td>
</tr>
<tr>
<td>3</td>
<td>2.1739 (20)</td>
<td>1.5851 (6)</td>
</tr>
<tr>
<td>4</td>
<td>1.8791 (18)</td>
<td>1.5679 (4)</td>
</tr>
<tr>
<td>5</td>
<td>1.9705 (19)</td>
<td>1.5638 (3)</td>
</tr>
<tr>
<td>6</td>
<td>1.7014 (12)</td>
<td>1.5776 (5)</td>
</tr>
<tr>
<td>7</td>
<td>1.7658 (14)</td>
<td>1.6363 (8)</td>
</tr>
<tr>
<td>8</td>
<td>1.7284 (13)</td>
<td>1.6686 (10)</td>
</tr>
<tr>
<td>9</td>
<td>1.8542 (16)</td>
<td>1.4809 (1)</td>
</tr>
<tr>
<td>10</td>
<td>1.7002 (11)</td>
<td>1.6680 (9)</td>
</tr>
</tbody>
</table>

In order to better illustrate that EOME can be able to distinguish these two groups (unhealthy and healthy samples), we use Mann-Whitney-\(U\) test to analyze the experimental results. The EOMEs of the 20 samples are shown in Table 3. We assume that there is no obvious difference between the two groups on the EOME. The test statistic \(U\) of two groups (\(u_1\) and \(u_2\) represent the \(U\) statistics of unhealthy and healthy samples, respectively) is calculated respectively, and the result is \(u_1 = 100 > u_2 = 0\). In the case of test level \(\alpha = 0.05\), we know \(U_{0.05,10} = 23\) by looking up the critical value of the \(U\) test form. Because the minimum value of \(u_1\) and \(u_2\) is less than the critical value, we reject the original hypothesis. Therefore, it can be considered that the two groups have obvious differences on EOME. In healthy samples, there are also some individuals...
whose EOME value is relatively large, which may be due to the difference of age, gender and living habits, etc. In conclusion, the measure EOME appears to be an effective tool for monitoring subtle changes in physiological structures. The advantages of those measures (EOME and RQA) lie in a high sensitivity and simple computation. Moreover, their calculations really do not require large amounts of data, and they are less affected by non-stationarity.

5. Conclusion

Recurrence quantity (RP), as a very powerful tool to study the characteristics of systems, has been applied to various fields. Many authors have studied the structural characteristics of RPs to reflect the nature of systems. To obtain some quantitative information, the recurrence quantification analysis (RQA) was invented. RQA measures allow us to identify different states of systems and the transitions to regular as well as other chaotic regimes in complex systems. These measures make us get more information about the system, even if they are only estimated by short and non-stationary time series.

RP is a symmetric and binary matrix, and its eigenvalues can reflect the basic characteristics of the complex systems. So, in this paper, we present a new measure based on recurrence matrix eigenvalues to quantify the dynamical properties of a given system. For a recurrence matrix, a part of its eigenvalues close to zero, and the absolute value of the other eigenvalues is relatively large for 0. The logarithmic treatment can better distinguish these eigenvalues, and these eigenvalues can be divided into three parts after logarithmic treatment. From the log eigenvalue spectrum we can qualitatively analyze the characteristics of the system. The greater the proportion of the middle portion of the log-eigenvalue spectrum, the lower the complexity of the system is. According to the eigenvalue spectrum, we divide the eigenvalues into three parts, and find out the probability distribution of the eigenvalues belongs to each part. In order to make a quantitative analysis, the EOME, the Shannon entropy of the eigenvalues of a recurrence matrix, is defined as a complexity measure of the systems. Entropy is a well-known measure of disorder, so we propose EOME and take it as a heuristic measure to detect the transitions between different system states.

Fig. 14. For the 20 samples, the results of RQA metrics and EOME are sorted from small to large and are shown in the picture above. In these four subplots, □ and * represent healthy and unhealthy samples, respectively. The fluctuation range of the experimental results of the RQA metrics is largely overlapping for healthy and unhealthy samples. We can find that EOME can well distinguish diseased individuals from these sample.
The EOME, as a metric based on the Shannon entropy, is expected to increase as given dynamical system changes from a non-chaotic to a chaotic regime. Firstly, we confirmed that EOME has the ability to distinguish different systems by comparing EOME of four characteristic typologies of RPs. The EOME of the periodic system is obviously smaller than that of the non-periodic systems, and small EOME value implies the system has a high predictability and low complexity. We illustrate the application of the EOME for the logistic map, and compare EOME with some RQA measures. The experiment shows that the measure EOME can distinguish different states of the systems as well as the transitions between different states, like chaos-period and chaos-chaos transitions. Moreover, there will be a rapid decrease and increase of EOME at the transitions between chaos and periodic windows, and maxima or peaks at the chaos-chaos transitions. For the application of EOME in practice, we apply EOME to physiology. EOME can successfully separate the ECG signals from healthy person and unhealthy person. All in all, EOME can be used as a new RQA measure to better identify dynamical transitions of complex systems represented by time series. Moreover, this measure does not require assumptions about the underlying dynamics, like stationarity or linearity.

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References


