RECURRENCE QUANTIFICATIONS: FEATURE EXTRATIONS FROM RECURRENCE PLOTS

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Recurrence plots are two-dimensional representations of multidimensional dynamics captured by applying time delays to a single series (vector) of ordinal data in time or space. Recurrence plots may be presented with beautiful lace-like structures, but most important is the inference these patterns make about the underlying dynamics. Complex patterns in recurrence plots can be reduced to primary diagonal, vertical and horizontal dot patterns aligned on a grid. It is the mixing and matching of these primary structures that give rise to all derivative graphical complexities. Once strict definitions are in place, features are easily quantified from recurrence plots of any form including cross recurrence plots.

Keywords: Recurrence quantification analysis; time series analysis; frequency domain analysis.

1. Introduction

Common human experience teaches that events and structures in the physical world possess and display properties of recurrence in time and space: the sun recurs daily; the full moon recurs monthly; wave patterns of the sea recur with repeating crest to crest intervals; stripes on zebras recur; heart beats and breathing cycles recur in time but with strong physiological state dependencies. Certainly the list of recurrent examples in nature is actually very large [Webber & Zbilut, 1998]. The fundamental idea is that once any pattern can be identified (like a quasi-template), if it is seen a second, third or more times, that pattern is said to be recurrent. Indeed, the whole scientific enterprise is critically dependent upon experimental results being repeatable. But unlike nature, strict recurrences of scientific results are not necessarily common.

The concept of recurrence has a long mathematical history [Poincaré, 1896; Feller, 1950; Kac, 1959], but interest in recurrence plots of multidimensional systems as a graphical were more recently renewed [Eckmann et al., 1987]. The multidimensionality was accomplished by time delaying a signal against itself [Takens, 1981]. Thus, instead of scoring for matching or recurrent points (scalars), points were plotted for matching or recurrent trajectories (vectors). By relaxing a radius parameter, the matches did not have to be exact. Because recurrences are tally counts (the most fundamental mathematical operation) instead of transformations, multidimensional recurrence plots
hold a strong attraction for physiologists who deal with systems that are characteristically nonlinear, nonstationary, punctuated by outliers, and blurred by noise. This is the very reason we got into the recurrence “game” over a decade ago. Our new contribution was to define specific features in recurrence plots, moving the interpretation of recurrence plots from the qualitative (subjective) to quantitative (objective) domain [Webber & Zbilut, 1994]. For a complete set of recurrence references, see http://www.recurrence-plot.tk/bibliography.php maintained by Marwan.

2. Recurrence Plot Features

Recurrence plots can be presented as lace-like structures, the features of which can be extracted as quantified as specific recurrence variables [Webber & Zbilut, 2005]. To introduce these quantifications, we first show the origination of four (finite) primary recurrent structures that can be projected upon a square matrix. Next we show how clumped primary structures result in numerous types of secondary recurrent structures. Then and only then are we prepared to define and use extracted features.
recurrence variables in meaningful and practical ways. The fundamental formula for the recurrence matrix is provided below and described in detail elsewhere [Eckmann et al., 1987; Marwan et al., 2003].

$$R_{i,j} = \Theta(\varepsilon - |\vec{x}_i - \vec{x}_j|) \quad i, j = 1, \ldots, N \quad (1)$$

### 2.1. Primary features

Dots in any two-dimensional matrix can be arranged in only four primary arrangements: diagonal lines; anti-diagonal lines; vertical lines; horizontal lines. For example, features such as checkmarks or crosshairs cannot be classified as primary structures because they are formed by the combination of two diagonals or two non-diagonals, respectively. The four types of primary structures are illustrated in the recurrence plot in Fig. 1 (blue) and were generated from a contrived time series consisting of 60 integer points in specific ordinal sequence (listed below and plotted in Fig. 1).

$$0,0,0,0,0,0,0,0,0,1,2,3,4,5,6,7,8,9,10,$$
$$9,8,7,6,5,4,3,2,1,0,2,4,6,8,10,10,10,10,10,$$
$$10,9,8,7,6,5,4,3,2,1,0,5,2,9,6,3,8,1,7,4$$

Specific points or spans of points in the time series are identified by ten letters (A-J) which result in the four primary recurrent structures when letters are paired. Thus, besides structureless, isolated points (JJ), there are diagonal lines (DH), anti-diagonal lines (BD, BH), vertical lines (CG), and horizontal lines (AE, AI) when the embedding dimension is one and the radius is zero. That is, recurrent points are scored only when exact integer matches occur in the time series (Fig. 1, blue). Increasing the embedding dimension to two while retaining the radius of zero removes all primary structures, save the diagonal lines (DH, Fig. 1, red). This means that there are no two-point vectors that exactly match each other except in the parallel trajectories of D and H in the time series. Although not demonstrated here, it is possible to still have primary vertical and horizontal line structures in recurrence plots when the embedding dimension exceeds one. On the other hand it can be shown that anti-diagonal lines can only be inscribed when the embedding dimension is one, but not larger. Anti-diagonal lines in higher dimensional recurrence plots are not to be viewed as primary structures, but fallouts of the combination of alternative primary structures into secondary structures (see Fig. 1).

### 2.2. Secondary features

By increasing the radius parameter from zero to one, the density of recurrent points increases as primary structures are thickened into secondary structures (Fig. 1, pink and green). This occurs because near-equal points are scored as matches (e.g. 1 recurs with 0, 1 and 2; 5 recurs with 4, 5 and 6; etc.). Even anti-diagonal lines widen when the embedding dimension is one, but they disappear altogether by increasing the embedding dimension to two.

Ubiquitous to all recurrence plots is the long central diagonal which results from points or vectors always matching themselves, independent of the embedding dimension and radius parameters selected. When strings of repetitive scalars occur in the time series (e.g. 0, 0, 0, 0, 0, 0, 0, 0, 0 for string A, Fig. 1), self-matches necessarily results in large black squares in the recurrence plot (e.g. AA and GG, Fig. 1). Similar rectangles (or squares) can also occur away from the central diagonal denoting singularities in the dynamic [Zbilut, 2004].

### 3. Quantifying Recurrence Plot Features

To date, seven useful quantitative features have been defined from qualitative recurrence plots [Webber & Zbilut, 1994; Marwan et al., 2002b], yet more may be forthcoming. These extracted values (termed recurrence variables) report on the same time series from seven different perspectives. Diagonal lines recurrences are “principal” of the four primary structures because they depict the presence of parallel trajectories along the time series of interest. For deterministic chaotic systems, diagonal lines recurrences are the specific instances when similar regions of the attractor are visited at different times for discrete [Hénon, 1976] as well as continuous [Rössler, 1976] mathematical systems. The longer the recurrent line, the longer the “visit”. In fact, periodic system such as sine waves display deterministic lines that stretch from boarder to boarder in the recurrence plot at an interval equal to the period of the signal [Webber & Zbilut, 1998]. It for this reason that diagonal lines in one way or another contribute to each of the seven quantifications now described. As a note, since recurrence plots are symmetrical across the central line of identity, computations are carried out only in the upper triangle, excluding the line of identity. For quantitative values, refer to Table 1 which reports the recurrence variables for the contrived time series.
Table 1. Recurrence variables of the ordinal sequence of points.

<table>
<thead>
<tr>
<th>M</th>
<th>RAD</th>
<th>REC</th>
<th>DET</th>
<th>LMAX</th>
<th>ENT</th>
<th>TND</th>
<th>LAM</th>
<th>TT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>9.266</td>
<td>45.732</td>
<td>11</td>
<td>3.093</td>
<td>-146.905</td>
<td>43.293</td>
<td>5.071</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>22.825</td>
<td>65.099</td>
<td>29</td>
<td>2.011</td>
<td>-167.017</td>
<td>62.624</td>
<td>3.373</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3.565</td>
<td>96.721</td>
<td>10</td>
<td>2.918</td>
<td>-188.542</td>
<td>80.328</td>
<td>4.455</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>6.780</td>
<td>83.621</td>
<td>11</td>
<td>2.551</td>
<td>-184.486</td>
<td>67.241</td>
<td>5.200</td>
</tr>
</tbody>
</table>

at various embedding dimensions (M) and radius values (RAD).

3.1. Recurrence (REC)

Percent recurrence is simply a quantitative measure of the density of recurrent points in the plot. REC can range from 0% (no recurrence) to 100% (full recurrence). Proper implementation of recurrence strategies insists that the recurrence matrix remain sparse (e.g. REC < 5%), else distant vectors would be scored as being close, falsely so.

\[
REC = \frac{1}{N^2} \sum_{i,j=1}^{N} R_{i,j}
\]  

3.2. Determinism (DET)

Percent determinism quantifies the fraction of recurrent points forming diagonal line structures. DET can range from 0% (no lines for stochastic systems) to > 0% (short lines for chaotic systems) to 100% (long lines for periodic systems).

\[
DET = \frac{\sum_{l=l_{\text{min}}}^{N} lP(l)}{\sum_{i,j}^{N} R_{i,j}}
\]

3.3. Divergence (DIV)

Divergence is a measure of how rapidly parallel trajectories diverge from one another. Divergence is best captured by the measurement of the most positive Lyapunov exponents [Wolf et al., 1985] and is characteristic of chaotic dynamics. It was posited that the longest diagonal line structure was inversely proportional to the most positive Lyapunov exponent [Eckmann et al., 1987]. This was confirmed to be the case for the logistic equation operating in its chaotic regime [Trulla et al., 1996]. Thus DIV is defined as the reciprocal of the longest line segment (L_{max}). The shorter the longest line is, the more divergent the trajectories.

\[
DIV = \frac{1}{L_{\text{max}}}
\]

\[
L_{\text{max}} = \max(\{l_i; i = 1, \ldots, N\})
\]

3.4. Entropy (ENT)

Entropy is defined as the Shannon information entropy of the line length distribution [Shannon, 1948]. That is, all diagonal lines lengths (integers) in the recurrence plot are distributed in a histogram from which bin nonzero probabilities, \(P(l)\), are computed and combined to obtain ENT as follows. ENT is a measure of system complexity and can range from 0 (no complexity) to ENT max (maximum complexity depending upon the number of nonzero bins, \(N\)).

\[
ENT = - \sum_{l=\text{line}}^{N} P(l) \ln P(l)
\]

3.5. Trend (TND)

Trend is a measure of recurrence plot homogeneity (TND ± 5) or heterogeneity (TND > 5; TND < 5) whence see [Webber et al., 1995] and is computed as the slope of the line of best fit through local percent recurrence values at integer distances away from the central diagonal. In this case, if the density of recurrent points remains near the same across the plot, TND will hover near a zero slope. If the density decreases away from the central diagonal, the TND will be strongly negative. If the density increases away from the central diagonal, the TND will be strongly positive. Near zero TNDs can be used to determine if the system is stationary in time, but nonzero TNDs can be used to access whether the system is on a transient.

\[
TREND = \sum_{i=1}^{N-2} \frac{[i - (N - 2)](REC_i - \langle REC_i \rangle)}{\sum_{i=1}^{N-2} \left[ i = \frac{N - 2}{2} \right]^2}
\]
3.6. Laminarity (LAM)

Laminarity scores for the fraction of recurrent points forming vertical line structures. The LAM name derives from the stacking of multiple parallel trajectories as lamina and also from stationarities in the time series (see Fig. 1). LAM can range from 0% (no laminarity) to 100% (full laminarity).

\[
LAM = \frac{1}{N} \sum_{\nu=v_{\min}}^{N} \nu P(\nu)
\]

3.7. Trapping time (TT)

Trapping time, the last recurrence variable, is computed as the average length of vertical line segments in the recurrent plot. TT is a measure of the time the dynamic remains trapped in a certain state and can be calibrated in time units if the digitization rate is known.

\[
TT = \frac{1}{N} \sum_{\nu=v_{\min}}^{N} \nu P(\nu) / \sum_{\nu=v_{\min}}^{N} P(\nu)
\]

4. Example Recurrence Quantifications

To illustrate the power and utility of recurrence quantifications on an actual physiological signal, we return to one of the earliest examples posited over a decade ago. Experiments were conducted to study the changing nature of the biceps electromyographic (EMG) recording during imposed isometric fatigue in the human [Webber et al., 1995]. The subject was seated while the biceps EMG was recorded during isometric (constant length) muscle loading. Light loads (1.4 kg) produced EMG signals of low amplitude; heavy loads (5.1 kg) produced EMG signals of high amplitude that could not be sustained much longer than two to three minutes (muscle fatigue).

![Recurrence plot of biceps EMG signal. The largest square (1.972 s) is partitioned into four smaller overlapping squares (1.024 s), each shifted in time (0.256 s).](image-url)
The EMG signal was subjected to spectral analysis using the fast Fourier transform (FFT) and recurrence quantification analysis (RQA) to study the sensitivity of both techniques to detect subtle EMG changes in the imposed time course of muscle fatigue.

The recurrence plot of the control period is shown in Fig. 2. The time series consists of 1972 points (0.032 s to 1.828 s) of the biceps EMG signal digitized at 1000 Hz and is plotted along both the horizontal and vertical axes of the plot. RQA parameters (also see below) were set as follows: delay = 4; embedding dimension = 10; norm = Euclidean norm; rescale = maximum distance; radius = 15%; line = 2. The large recurrence plot shows recurrent points homogeneously distributed

![Plot](image)

Fig. 3. Quantitative analysis of the fatiguing biceps muscle. (a) EMG of biceps. (b) FFT center frequency. (c) RQA deter-

minism. The dashed line marks the increase of the muscle load from 1.4 kg to 5.1 kg. The black dots indicate differential (RQA versus FFT) detections of fatigue onset.
over the sparse matrix. The seven corresponding RQA variables are quantified as follows: \( \text{REC} = 0.335; \text{DET} = 60.905; \text{LMAX} = 118; \text{ENT} = 1.711; \text{TND} = -0.129; \text{LAM} = 55.660; \text{TT} = 2.501. \)

Partitioning the recurrence plot into four smaller overlapping squares (epochs), each representing 1024 points (1.024 s) of the time series (Fig. 2), introduces how recurrence variables can be followed over time. For example, each smaller square, shifted in time, has its own unique constellation of recurrence variables. Figure 3 plots DET from RQA and center frequency from FFT analyses, paralleled with the EMG time signal itself. During the light-load control, the FFT center frequency averaged about 70 Hz while the recurrence DET averaged about 70%. Moving from window to window, both measured variables showed fluctuations which were smoothed by running a 10-degree polynomial through the data. Not shown are the fluctuations of REC which varied from 0.079% to 7.618% during the fatiguing period. The key points of these results are that when the heavy weight was applied, the DET variable increased at a faster rate than the center frequency variable decreased. And within a group of ten tested subjects, the sensitivity of RQA was shown to be significantly higher \((P < 0.05)\) than the sensitivity of FFT. It was concluded that the multidimensional, nonlinear technique (RQA) pulled out details from the signal unavailable to the single-dimensional, linear technique (FFT).

5. Cross Recurrence Quantifications

Another useful application of recurrence quantifications is the extraction of features derived from cross-recurrence plots. In this case and similar to cross correlations, recurrence plots are generated from paired input signals, revealing subtle nonlinear interactions [Zbilut et al., 1998; Marwan et al., 2002a]. Take for example the following coupled oscillator experiment [Shockley et al., 2002]. A tray was filled with viscous fluid (nonlinear coupler) and pushed and pulled back and forth by a sinusoidal motor. Into this tray was submersed a rotor blade which was set into rotary motion by a gravity driven weight/pulley system. The complex coupling of the rotor (follower) to the tray (driver) depended upon the viscosity of the coupling fluid. In Fig. 4 the cross-recurrence plot of a low-viscosity coupling between rotor and tray is

![Cross-recurrence plot](image)
plotted using the following parameters: delay = 1; embedding dimension = 5; norm = Euclidean norm; rescale = maximum distance; radius = 2%; line = 2. The recurrent point patterns represent the nonlinear, nonperiodic nature of the coupling as best captured in the extracted recurrence variables: \%REC = 1.480; \%DET = 94.103; LMAX = 24; ENT = 3.041; TND = −0.813; \%LAM = 74.834; TT = 3.300. By changing the viscosity of the coupling fluid, the subtle changes in the rotor-tray interactions were quantified by changes in the recurrence variables (Table 2, average of four runs each).

From the cross-recurrence plot, it is possible to compute the nonlinear spectrum of recurrence intervals and make important comparisons with linear FFT spectrum. To do this, the vertical time intervals between all recurrent points are scored, converted to time units (based upon the digitization frequency), and binned within a histogram. By taking the reciprocal of the histogram time intervals (e.g. sec), the histogram bins are conveniently recalibrated in frequency units (e.g. Hz). The results are plotted in Fig. 5. In the top panel, the standard linear FFT spectrum of the rotor is plotted. One frequency peak is centered on the driver frequency (0.617 Hz) and a second higher frequency peak (1.3 Hz) expresses the interaction frequency of the coupled oscillators. In the bottom panel, the recurrence spectrum of the paired rotor-driver system is plotted. Again, part of the spectrum shows the coupling synchronized at the driver frequency. But the highly nonlinear interactions are splayed out over a wider, higher frequency band (1.0 to 1.7 Hz). Not only is the recurrence spectrum of higher resolution than the FFT spectrum, but it also reveals subtle nonlinearities lost in the linear transformations.

6. Conclusions

Recurrence quantifications are now in their second decade of implementation across many disciplines of study [Marwan, 2005]. RQA continues to be shown to be a powerful tool in the diagnosis of complex dynamical systems that are high-dimensional, nonlinear, and noisy. That being stated, however, RQA is definitely not a stand-alone technique. RQA has been coupled with principle component analysis, for example, to give even deeper insights into dynamical systems [Zbilut et al., 1998].

While recurrence plots are very easy to generate due to conveniently available software packages [Kononov, 2004; Marwan, 2005; Webber, 2007], three things should be remembered. First, the selection of recurrence parameters should be approached reasonably. It is unreasonable to expect all parameters to have optimal settings, especially when dealing with nonstationary systems. Second, the meaning of recurrence quantifications should be understood carefully. The details are in the numbers, not the plots; much can be learned from studying “toy” signals. Third, the interpretation of recurrence quantifications should be conservative. It is always wise to exercise caution and arrive at conservative conclusions.

Recurrence analysis continues to progress as investigators keep contributing to the field. For example, one new avenue of research may be in the recurrence plot analysis of nonstationary
data: e.g. the call of the gibbon [Facchini et al., 2005]. These types of plots display complicated patterns, begging the issue of whether new recurrence quantifications might trigger off from these tertiary structures, not just the primary and secondary line structures described in this paper.

References


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