Experimental distinction between chaotic and strange nonchaotic attractors on the basis of consistency

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Strange nonchaotic attractors (SNAs) often appear in dynamical systems driven by a quasiperiodic signal. An SNA is a geometrically strange attractor for which typical orbits have nonpositive Lyapunov exponents. In fact, typical orbits are insensitive to initial conditions under a common quasiperiodic signal. Owing to their characteristics reminiscent of both quasiperiodic order and chaos, SNAs have attracted the attention of researchers from both theoretical and experimental perspectives. Conventionally, the information dimension is often employed to distinguish between SNAs and chaotic attractors in experiments. However, using the information dimension for the experimental distinction between these types of attractors is difficult as the estimation of the information dimension is highly sensitive to noise, and requires rather long time series. Recently, Ngamga et al. proposed a method for distinguishing between SNAs and chaotic attractors, which utilizes the consistency property of SNAs. Consistency here means that a nonlinear system shows the same response to the same input signal after some transient time, regardless of the initial conditions. In the method proposed by Ngamga et al., the determinism for a cross-recurrence plot of two response time series to the same input signal is used as a consistency measure. However, to our knowledge, there is no research paper on using their cross-recurrence analysis for distinguishing between SNAs and chaotic attractors. In this study, we evaluated the consistency of a system by using the zero-delay normalized cross-correlation and Ngamga et al.’s cross-recurrence methods. By combining spectrum analysis and evaluation of consistency, we showed that SNAs and chaotic attractors can be distinguished more precisely as compared to conventional methods.

I. INTRODUCTION

SNAs, which have a fractal structure and a non-positive largest Lyapunov exponent, appear in quasiperiodically forced dynamical systems. Thus far, SNAs have been studied in various nonlinear models and have been observed in experiments using electronic circuits, magnetoelastic ribbons, and electrochemical cells (see also Refs. 28–30, and references therein). However, the experimental distinction between SNAs and chaotic attractors is not straightforward since standard time series analysis methods cannot provide a valid estimate for the largest Lyapunov exponent of strange nonchaotic attractors. Therefore, it is important to establish an analysis method of experimental data for distinguishing between SNAs and chaotic attractors.

After some transient time and regardless of the initial conditions, various nonlinear systems produce the same response when the same input signal is repeatedly presented, and such reproducibility of response is called consistency. If a system has an SNA as a unique attractor, the system can produce a consistent response to a repeated quasiperiodic signal since the Lyapunov exponents corresponding to the response system are all negative. On the other hand, if a system is chaotic, the consistency is usually low.
distinguished using their cross-recurrence method. Therefore, in this paper, we demonstrate analysis of experimental data for distinguishing between SNAs and chaotic attractors based on Ngamga et al.’s cross-recurrence method. We also use the usual consistency measure, namely the zero-delay normalized cross-correlation. We conducted experiments using a chaotic neuron integrated circuit\(^\text{35}\) and observed quasiperiodic, strange nonchaotic, and chaotic attractors. By combining the spectrum analysis and the evaluation of consistency, we show that SNAs and chaotic attractors can be distinguished more precisely as compared to conventional methods.

This paper is organized as follows. In Sec. II, we present a chaotic neuron integrated circuit used in the experiments, and a chaotic neuron model\(^\text{36}\) with a quasiperiodic external force. Methods for analysis of time series are explained in Sec. III. We show the results of experiments and analysis in Sec. IV, and conclude this paper in Sec. VI.

II. CIRCUIT IMPLEMENTATION AND EXPERIMENTAL CONDITIONS

We conducted experiments using a chaotic neuron integrated circuit\(^\text{35}\) that implements a chaotic neuron model\(^\text{36}\). The circuit is shown in Fig. 1. The operation of the circuit is organized by the signals of the non-overlapping clocks \(\phi B, \phi C, \) and \(\phi D\). The voltages \(\zeta(t+1), \xi(t+1), y(t+1), \) and \(x(t+1)\) in Fig. 1 satisfy the following equations:

\[
\begin{align*}
\zeta(t+1) &= k_r \zeta(t) + \alpha x(t) + a, \quad (1) \\
\xi(t+1) &= b \cos(2\pi \theta(t+1)), \quad (2) \\
y(t+1) &= \zeta(t+1) + \xi(t+1), \quad (3) \\
x(t+1) &= f(y(t+1)), \quad (4) \\
\theta(t+1) &= \theta(t) + \omega \mod 1, \quad (5)
\end{align*}
\]

where \(t\) the discrete-time, \(\zeta(t)\) the internal state of the model neuron, \(k_r\) the decay coefficient of the internal state, \(\alpha\) a scaling parameter for the refractoriness, \(a\) a constant bias on the system, \(\zeta(t)\) the quasiperiodic external force, \(y(t)\) the sum of \(\zeta(t)\) and \(\xi(t)\), \(x(t)\) the output of the neuron, and \(\theta(t)\) the phase. Equations (2)–(5) can be reduced to the following two-dimensional map:

\[
\begin{align*}
\zeta(t+1) &= k_r \zeta(t) + \alpha f(y(t)) + b \cos(2\pi \theta(t)) + a, \\
\theta(t+1) &= \theta(t) + \omega \mod 1,
\end{align*}
\]

where \(f(\cdot)\) is a monotonically decreasing nonlinear output function. The input-output characteristics of \(f(\cdot)\) can be controlled through the external voltages \(V_{IN}\) and \(V_{RP}\).\(^\text{35}\) We set \(V_{IN} = 1.6\) V, and \(V_{RP} = -1.6\) V in the experiments. The parameter \(b\) is the amplitude of the quasiperiodic force \(\xi(t)\), and \(\omega = (\sqrt{5} - 1)/2\). We used an analog input/output board PXI-6289 (National Instruments) to input the quasiperiodic external force \(\zeta(t)\) into the circuit and to measure the output time series. The difference between the previous experimental setting in Ref. 35 and the present one is that we used the quasiperiodic external force to drive the chaotic neuron integrated circuit.

The parameters \(k_r, \alpha, \text{ and } a\) are generated by an 8-bit programmable capacitive array (PCA), and their values are given by the following capacitance ratios: \(k_r = 1 - C_k/C_\zeta, \alpha = C_\alpha/C_\zeta, \text{ and } a = (C_a/C_\zeta) V_{BIAS}\). In the circuit experiments, we fixed the values of the capacitors as \(C_\zeta = C_x = C_e = 12.7\) pF, \(C_k = 10.6\) pF, \(C_\alpha = 8.2\) pF, \(C_a = 2.6\) pF, \(C_p = C_{scale} = 2.5\) pF, and \(V_{BIAS} = 1.65\) V. The amplitude \(b\) was varied in the range of \(0 \leq b \leq 0.2\) V in steps of 0.0025 V.

We measured the input-output characteristic \(f(y)\) generated by the nonlinear output function circuit, and approximated the characteristic \(f(y)\) by a cubic spline interpolation (see Appendix for details about the interpolation). The shape of \(f(y)\) is shown in Fig. 2. We can obtain good agreements between numerical and experimental attractors by using a precisely approximated input-output characteristic expressed by \(f(y)\).

![Fig. 1. Schematic of the chaotic neuron integrated circuit.](image)

The operation of the circuit is organized by the signals of the non-overlapping clocks \(\phi B, \phi C, \) and \(\phi D\).
Attractors are classified as chaotic if the case of SNAs, which are non-smooth. The exponents otherwise.

III. ANALYSIS METHODS

A. Numerical model analysis

1. Lyapunov exponent \( \hat{\lambda} \)

The dynamics of Eq. (6) is characterized by two Lyapunov exponents denoted as \( \{ \hat{\lambda}_i, \lambda_i \} \). One of the Lyapunov exponents (e.g., \( \lambda_i \)) is always zero owing to the rigid rotation of phase \( \theta(t) \). The Lyapunov exponent corresponding to the direction of \( \zeta \) is given by

\[
\hat{\lambda}_\zeta = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \log \left| \frac{\partial \zeta(t+1)}{\partial \zeta(t)} \right|.
\] (7)

Attractors are classified as chaotic if \( \hat{\lambda}_\zeta > 0 \) and nonchaotic otherwise.

2. Phase sensitivity exponent \( \mu \)

The system (6) exhibits quasiperiodic attractors, SNAs, and chaotic attractors. A quasiperiodic attractor is represented by a torus. In order to classify tori and SNAs in the circuit model, we use the phase sensitivity exponent \( \mu \), defined as follows:37

\[
\Gamma(t) = \min_{\zeta(0), \theta(0)} \max_{1 \leq m \leq t} \left| \frac{\partial \zeta(m)}{\partial \theta(0)} \right| \simeq \tau^\mu.
\] (8)

The phase sensitivity function \( \Gamma(t) \) is bounded for tori because \( \mu = 0 \). In contrast, \( \Gamma(t) \) grows with \( \tau \) due to \( \mu > 0 \) in the case of SNAs, which are non-smooth. The exponents \( \hat{\lambda}_\zeta \) and \( \mu \) divide a parameter space into torus, SNA, and chaos regions.

B. Conventional time series analysis

1. Information dimension \( D_1 \)

In several experimental studies,23,24,26 the information dimension \( D_1 \) has been used to distinguish between SNAs and chaotic attractors since standard time series analysis methods used for estimating Lyapunov exponents do not produce reliable results for SNAs.31,32 The information dimension of an attractor is defined as

\[
D_1 = \lim_{r \to 0} \frac{\log \frac{M(r)}{\log_2 1/r}}{\log_2 1/r},
\] (9)

where \( r \) is the edge length of the boxes covering the attractor. Here, the information \( H(r) \) is given by

\[
H(r) = \sum_{i=1}^{M(r)} p_i(r) \log_2 p_i(r),
\]

where \( p_i(r) \) is the probability at which a typical trajectory visits the \( i \)th box, and \( M(r) \) is the minimum number of boxes required to cover the attractor.

For the map shown in Eq. (6), \( D_1 \) cannot be less than one due to the rigid rotation of \( \theta(t) \). It has been proven that \( D_1 \) is bounded from above by the Lyapunov dimension \( D_L \). For SNAs with one zero and one negative Lyapunov exponent, \( D_L = 1 \) and, therefore, \( D_1 = 1 \).8,19 This prediction coincides with the Kaplan-Yorke conjecture39 such that \( D_1 = D_L \) for typical attractors. On the other hand, for chaotic attractors with one zero and one positive Lyapunov exponent, \( D_L = 2 \) and we estimate \( D_1 = 2 \) according to the Kaplan-Yorke conjecture.

However, the experimental distinction between SNAs and chaotic attractors using the information dimension is difficult as \( D_1 \) is sensitive to noise, and its calculation requires a large amount of time series data. Moreover, a numerical study confirmed that the value of information dimension changes little around a transition point from SNAs to chaotic attractors in either presence or absence of dynamical noise.40

2. Spectral distribution function \( N(\varepsilon) \)

The spectral distribution function \( N(\varepsilon) \) is defined as the number of spectral components larger than a threshold \( \varepsilon \). Function \( N(\varepsilon) \) will be \( N(\varepsilon) \sim \ln \varepsilon \) for a two-frequency torus and \( N(\varepsilon) \sim (\ln 1/\varepsilon)^{1/2} \) for a three-frequency torus.20 Only the case of a two-frequency torus is considered in this paper since no three-frequency torus appear in the model. For SNAs, the spectral distribution function \( N(\varepsilon) \) behaves like \( N(\varepsilon) \sim \varepsilon^{-\beta} \) with \( 1 < \beta < 2 \).20

C. Consistency based time series analysis

1. Zero-delay normalized cross-correlation

The zero-delay normalized cross-correlation \( CC \) is defined as follows:

\[
CC = \frac{\langle \hat{X}_t \hat{Y}_t \rangle}{\sqrt{\langle \hat{X}_t^2 \rangle \langle \hat{Y}_t^2 \rangle}},
\] (10)

where \( X \) and \( Y \) are two different time series, \( \langle \cdot \rangle \) is a long time average with respect to discrete-time \( t \), \( \hat{X}_t = X_t - \langle X_t \rangle \), and \( \hat{Y}_t = Y_t - \langle Y_t \rangle \). We have \( CC = 1 \) for the case of perfect consistency \( X_t = Y_t \).

2. Determinism of a cross-recurrence plot \( D_{ET} \)

We calculate the cross-recurrence matrix41 to quantify the consistency as follows:

\[
CR_t = \Theta(\delta - || \hat{X}_t - \hat{Y}_{t'} ||) \ (t, t' = 1, 2, ..., N)
\] (11)

of separate trajectories \( \hat{X}_t \) and \( \hat{Y}_{t'} \), where \( \Theta(\cdot) \) is the Heaviside function, \( t \) and \( t' \) denote discrete–times, and \( \delta \) is a
TABLE I. Summary of classification indexes for the attractors.

<table>
<thead>
<tr>
<th>Attractor</th>
<th>Characteristic exponents</th>
<th>Conventional method (analysis based on information dimension)</th>
<th>Analysis based on consistency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-freq. torus</td>
<td>( \lambda_c &lt; 0, \mu = 0 )</td>
<td>( D_t = 1, N(\varepsilon) \sim \ln \varepsilon )</td>
<td>( CC = 1, D_{ET} = 1 )</td>
</tr>
<tr>
<td>SNA</td>
<td>( \lambda_c &lt; 0, \mu &gt; 0 )</td>
<td>( D_t = 1, N(\varepsilon) \sim \varepsilon^{-\beta} (1 &lt; \beta &lt; 2) )</td>
<td>( CC = 1, D_{ET} = 1 )</td>
</tr>
<tr>
<td>Chaos</td>
<td>( \lambda_c &gt; 0 ) (( \mu ) undefined)</td>
<td>( D_t = 2 )</td>
<td>( CC &lt; 1, D_{ET} &lt; 1 )</td>
</tr>
</tbody>
</table>

\( \lambda_c \): Lyapunov exponent, \( \mu \): phase sensitivity exponent, \( D_t \): information dimension, \( N(\varepsilon) \): spectral distribution function, \( CC \): zero-delay normalized cross-correlation, and \( D_{ET} \): determinism of cross-recurrence plot.

threshold. The threshold \( \delta \) was set to 0.2 times the average of the standard deviation of the two trajectories, \( \tilde{x}_i \) and \( \tilde{y}_i \), according to Ref. 1. The main diagonal line \( (t = \tilde{t}) \) of the cross-recurrence plot will be a continuous line if the system exhibits perfect consistency, and discontinuous otherwise.\(^1\) The length \( l \) of the continuous component of the diagonal line in a cross-recurrence plot indicates the time period during which two trajectories stay within a neighborhood of \( \delta \) of each other. The determinism \( D_{ET} \) (Refs. 1 and 42) is given by

\[
D_{ET} = \frac{\sum_{l=1}^{N} ID(l)}{\sum_{l=1}^{N} lD(l)},
\]

where \( D(l) \) is the frequency distribution of diagonal line length \( l \) in the main diagonal line. We set \( l_{\text{min}} = 2 \) in this paper.

In the case of chaotic attractors, the distance between nearby trajectories increases exponentially. Therefore, the value of \( D_{ET} \) will be small. In contrast, for tori and SNAs, which have a negative Lyapunov exponent \( \lambda_c \), the determinism \( D_{ET} \) is unity since identical time series are obtained for the same quasiperiodic external force, after some transients. Ngamga et al.'s cross-recurrence method\(^4\) is more robust with respect to a small amount of noise than the zero-delay normalized cross-correlation since orbital differences smaller than the threshold do not affect the value of \( D_{ET} \). The computation of \( CC \) and \( D_{ET} \) does not require long time series, but it does require the response time series to be measured at least twice for the same input signal.

Moreover, in order to evaluate the influence of noise on the consistency, we conducted numerical experiments by adding Gaussian random noise \( \psi(t) \) to the dynamics \( \zeta(t) \) as follows:

\[
\zeta(t+1) = k_c \zeta(t) + 2f(\zeta(t) + b \cos(2\pi \theta(t))) + a + \psi(t),
\]

where the average of \( \psi(t) \) is zero, and we use three levels of standard deviation: \( \rho = 1 \times 10^{-5} \text{V}, 3 \times 10^{-5} \text{V}, \text{and} 5 \times 10^{-5} \text{V} \). We summarize the classification indexes of attractors and analysis methods in Table I.

IV. ANALYSIS RESULTS

A. Numerical analysis

1. Transition of attractors

We show attractors for the numerical model for different values of parameter \( b \) in Figs. 3(a)–3(c). These attractors are characterized by the Lyapunov exponents: \( \lambda_c = 0.151 \) for
transition point \( b_c \) \((\approx 0.0575)\) mentioned above. In the presence of noise, \( CC \) becomes less than unity even for tori and SNAs, as shown in Fig. 5(a). The inset in Fig. 5(a) shows that \( CC \) decreases as \( \rho \) increases. Thus, the transition point from SNAs to chaos is obscured by noise. However, the drastic decrease in \( CC \) strongly suggests the existence of a transition from SNAs to chaos.

As shown in Fig. 5(b), the dependence of \( D_{ET} \) on \( b \) is qualitatively the same as that for \( CC \). That is, \( D_{ET} = 1 \) for tori and SNAs \((b > b_c)\), and \( D_{ET} < 1 \) for chaotic attractors \((b < b_c)\). Again, the effect of noise \( \psi(t) \) slightly obscured the transition point from SNAs to chaos, but the drastic decrease in \( D_{ET} \) strongly indicates the existence of the transition.

### B. Experimental analysis

#### 1. Analysis based on information dimension \( D_1 \) and spectral distribution function \( N(\epsilon) \)

The attractors obtained from the circuit in Fig. 1 are shown in Figs. 3(d)–3(f). The information dimensions \( D_1 \) of these attractors are shown in Table II. They were calculated using time series of \( 10^6 \) data points and were the results of least squares fitting to the plots of \( H(r) \) vs. \( \log_2(1/r) \) within the range \( 3.46 \leq \log_2(1/r) \leq 6.19 \).

The information dimension of the attractor in Fig. 3(d) \((b = 0.05 \text{ V})\) is \( D_1 = 1.85 \), which is somewhat close to two. On the other hand, \( D_1 = 1.24 \) and \( D_1 = 1.09 \) for the attractors in Figs. 3(e) \((b = 0.13 \text{ V})\) and 3(f) \((b = 0.5 \text{ V})\), respectively, which are somewhat close to one. Therefore, the attractor shown in Fig. 3(d) can be considered to be chaotic, and the attractors shown in Figs. 3(e) and 3(f) can be considered to be nonchaotic. However, these estimated values of \( D_1 \) show non-negligible deviations from the theoretical values: \( D_1 = 2 \) for a chaotic attractor; and \( D_1 = 1 \) for a nonchaotic attractor. Thus, the classification of attractors according to the value of \( D_1 \) in this experiment is inconclusive.

Figure 6(a) shows the respective spectral distribution functions \( N(\epsilon) \) for the two attractors in Figs. 3(b) and 3(e). The scaling \( N(\epsilon) \sim \epsilon^{-\beta} \) is observed, where the scaling exponent \( \beta = 1.38 \) for the attractor in Fig. 3(b) (model) and \( \beta = 1.13 \) for the attractor in Fig. 3(e) (circuit). The scaling with \( 1 < \beta < 2 \) is characteristic of SNAs. In contrast, the spectra in Fig. 6(b) for the attractors Figs. 3(c) (model) and 3(f) (circuit) obey the scaling relationship of quasiperiodic attractors, where \( N(\epsilon) \sim \ln(\epsilon) \).

To overcome the difficulties associated with distinguishing between SNAs and chaotic attractors by the use of information dimension, we also performed analysis based on the consistency, and the results are presented in Subsection IV B 2.

#### 2. Analysis based on consistency and spectral distribution function \( N(\epsilon) \)

Figures 7(a)–7(e) show pairs of time series \( \zeta(t) \) under the same quasiperiodic force, which start from different initial conditions \( \zeta(0) \), for the attractors shown in Figs. 3(d) \((b = 0.05 \text{ V})\), 3(e) \((b = 0.13 \text{ V})\), and 3(f) \((b = 0.5 \text{ V})\), respectively. The two time series are desynchronized, almost synchronized, and completely synchronized in Figs.
7(a)–7(c), respectively. These results indicate that the consistency is low for the attractor in Fig. 3(d) and high for the attractors in Figs. 3(e) and 3(f).

Table II shows the values of $CC$ and $DET$ for the attractors in Figs. 3(d)–3(f). We calculated each values of $CC$ and $DET$ as the ensemble average for 45 possible combinations from 10 time series for different initial conditions $f(0)$ under the same quasiperiodic external force. Each time series consists of 20,000 data points. The attractor shown in Fig. 3(d) ($b = 0.05$ V) exhibits low consistency with $DET/C^2$ and $CC/C^2$. Therefore, this attractor is considered to be chaotic.

For the attractor shown in Fig. 3(e) ($b = 0.13$ V), both $CC$ and $DET$ are close to unity. Moreover, this attractor exhibits the spectral distribution function of SNAs as described in Sec. IV B 1 (see Fig. 6(a)). Therefore, this attractor should be an SNA. This attractor does not show perfect consistency, but this may be due to noise, as demonstrated in the numerical experiments. On the other hand, the attractor shown in Fig. 3(f) ($b = 0.5$ V) exhibits perfect consistency and has a spectral distribution function of a torus (see Fig. 6(b)). Thus, it is concluded that this attractor is a torus.

The consistency measures $CC$ and $DET$ as functions of $b$ are shown in Figs. 5(c) and 5(d), respectively. We can see a drastic decrease in $CC$ and $DET$ in the interval $0.0575 < b < 0.075$ from SNAs to chaos is shown hatched in (c) and (d).

The calculation of $DET$ for the attractors in Figs. 3(d)–3(f). This table shows measurements results.

<table>
<thead>
<tr>
<th>Attractors</th>
<th>$D_1$</th>
<th>$N(e)$</th>
<th>$CC$</th>
<th>$DET$</th>
<th>Estimated types</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 3(d)</td>
<td>1.85 ± 0.02</td>
<td>0.244</td>
<td>0.893</td>
<td>Chaos</td>
<td></td>
</tr>
<tr>
<td>$b = 0.05$ V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fig. 3(e)</td>
<td>1.24 ± 0.03</td>
<td>$N(e) \sim e^{-\beta}$, $1 &lt; \beta = 1.13 &lt; 2$</td>
<td>0.901</td>
<td>0.994</td>
<td>SNA</td>
</tr>
<tr>
<td>$b = 0.13$ V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fig. 3(f)</td>
<td>1.09 ± 0.02</td>
<td>$N(e) \sim \ln(e)$</td>
<td>1.000</td>
<td>1.000</td>
<td>Two-freq. torus</td>
</tr>
<tr>
<td>$b = 0.5$ V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

V. DISCUSSION

Let us discuss the dependence of $DET$ on the value of $l_{\text{min}}$. Figure 8 shows the determinism $DET$ as a function of $b$ for...
different values of $l_{\text{min}}$. When the noise does not exist, the
determinism $D_{ET}$ is unity for nonchaotic attractors regardless of
$l_{\text{min}}$, and decreases for chaotic attractors as $l_{\text{min}}$ increases
(see Fig. 8(a)). On the other hand, when the noise exists, the
determinism $D_{ET}$ decreases around transition regions even for
SNAs as $l_{\text{min}}$ increases (see Fig. 8(b) and 8(c)). Therefore,
$l_{\text{min}} = 2$ gives the value of $D_{ET}$ closest to the theoretical value
for SNAs and is most suitable when we distinguish SNAs
from chaotic attractors based on the value of $D_{ET}$.

VI. CONCLUSION

Experimental distinction between SNAs and chaotic
attractors has been considered a difficult problem because
In a previous study, Ngamga et al. proposed the cross-recurrence method to distinguish between SNAs and chaotic attractors based on the consistency property of SNAs. However, to the best of our knowledge, their cross-recurrence method has not been applied for real experimental data. It is worthy to test their method in real experiments because experiments are sometimes sensitive to experimental conditions, dynamical, and/or observational noises. Therefore, we conducted an observation experiment of these attractors on the basis of the consistency, using the chaotic neuron integrated circuit driven by the quasiperiodic external force.

In the experiment, tori, SNAs, and chaotic attractors were observed. We calculated the consistency of these attractors by using both Ngamga et al.’s cross-recurrence and the zero-delay normalized cross-correlation methods.

We distinguished between tori, SNAs, and chaotic attractors by evaluating the consistency and spectral distribution function experimentally for the first time in this paper. Moreover, it was shown that the determinism $D_{ET}$ is more robust to noise than the cross-correlation $CC$ since the values of $D_{ET}$ is closer to the theoretical value 1 than those of $CC$ in the SNA regime when the noise exists (see the SNAs region of Figs. 5(a) and 5(b), and the values of $CC$ and $D_{ET}$ in Table II). From our results shown in Table II, we conclude that the consistency (in particular $D_{ET}$) is more suitable than the conventionally used information dimension to experimentally distinguish between SNAs and chaotic attractors when we can reproduce the same input signal.

The value of $D_{ET}$ is slightly smaller than the theoretical value 1 for SNAs possibly because of the observational and/or dynamical noise. However, a drastic decrease of consistency measures with a change of parameter values suggests the existence of a transition region from SNAs to chaotic attractors.

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**APPENDIX: DETAILS ON THE NONLINEAR OUTPUT FUNCTION $f(\cdot)$**

The nonlinear output function $f(\cdot)$ is generated in a chaotic neuron integrated circuit by using nonlinear physical properties of MOSFETs, and its form cannot be expressed analytically. Therefore, in this study, we approximated the measured input-output characteristic by using a cubic spline interpolation. The coefficients for the interpolation are shown in Table III.

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**TABLE III. Coefficients of cubic spline interpolation.**

<table>
<thead>
<tr>
<th>Ranges</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.433 \leq y &lt; -0.9313$</td>
<td>$-0.05790$</td>
<td>$0.000$</td>
<td>$-0.08297$</td>
<td>$1.420$</td>
<td>$-1.443$</td>
</tr>
<tr>
<td>$-0.9313 \leq y &lt; -0.6236$</td>
<td>$-0.4604$</td>
<td>$-0.08884$</td>
<td>$-0.1284$</td>
<td>$1.370$</td>
<td>$-0.9313$</td>
</tr>
<tr>
<td>$-0.6236 \leq y &lt; -0.3889$</td>
<td>$-1.167$</td>
<td>$-0.4604$</td>
<td>$-0.2974$</td>
<td>$1.310$</td>
<td>$-0.6236$</td>
</tr>
<tr>
<td>$-0.3889 \leq y &lt; -0.2848$</td>
<td>$-2.026$</td>
<td>$-1.282$</td>
<td>$-0.7063$</td>
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<td>$-0.2848 \leq y &lt; -0.1767$</td>
<td>$-27.12$</td>
<td>$-1.915$</td>
<td>$-1.039$</td>
<td>$1.110$</td>
<td>$-0.2848$</td>
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<tr>
<td>$-0.1767 \leq y &lt; -0.1321$</td>
<td>$-17.97$</td>
<td>$-10.71$</td>
<td>$-2.403$</td>
<td>$0.9413$</td>
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<td>$-0.1321 \leq y &lt; -0.08171$</td>
<td>$-16.79$</td>
<td>$-13.11$</td>
<td>$-3.465$</td>
<td>$0.8113$</td>
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<tr>
<td>$-0.08171 \leq y &lt; -0.01066$</td>
<td>$33.75$</td>
<td>$-15.65$</td>
<td>$-4.914$</td>
<td>$0.6012$</td>
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<tr>
<td>$-0.01066 \leq y &lt; 0.001477$</td>
<td>$-353.9$</td>
<td>$-8.454$</td>
<td>$-6.627$</td>
<td>$0.1851$</td>
<td>$-0.01066$</td>
</tr>
<tr>
<td>$0.001477 \leq y &lt; 0.03148$</td>
<td>$69.07$</td>
<td>$-21.34$</td>
<td>$-6.989$</td>
<td>$0.1028$</td>
<td>$0.001477$</td>
</tr>
<tr>
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<td>$324.9$</td>
<td>$-15.12$</td>
<td>$-8.083$</td>
<td>$-0.1242$</td>
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<td>$0.08106 \leq y &lt; 0.1414$</td>
<td>$-68.25$</td>
<td>$33.20$</td>
<td>$-7.187$</td>
<td>$-0.5225$</td>
<td>$0.08106$</td>
</tr>
<tr>
<td>$0.1414 \leq y &lt; 0.2606$</td>
<td>$-52.52$</td>
<td>$20.85$</td>
<td>$-3.927$</td>
<td>$-0.8501$</td>
<td>$0.1414$</td>
</tr>
<tr>
<td>$0.2606 \leq y &lt; 0.6058$</td>
<td>$-1.475$</td>
<td>$2.065$</td>
<td>$-1.196$</td>
<td>$-1.111$</td>
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</tr>
<tr>
<td>$0.6058 \leq y &lt; 1.091$</td>
<td>$-0.3852$</td>
<td>$0.5374$</td>
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<td>$-1.338$</td>
<td>$0.6058$</td>
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<tr>
<td>$1.091 \leq y \leq 1.587$</td>
<td>$-0.01584$</td>
<td>$-0.02357$</td>
<td>$-0.04774$</td>
<td>$-1.400$</td>
<td>$1.091$</td>
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