On the unique reconstruction of a signal from its unthresholded recurrence plot

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We address the information content of unthresholded recurrence plots, generated by the time-delay embedding method from scalar signals admitting a Fourier series representation (including periodic and sampled signals). This is important for making valid inferences from unthresholded recurrence plots. A graph theoretic framework is developed to give a complete analysis of the impact of the choice of time-delay and embedding dimension on information content. A distance measure for unthresholded recurrence plots is introduced to approach signal reconstruction and approximation by minimization, robust to inaccuracies and noise. Examples and an application from EEG analysis clarify the theoretical results and demonstrate their practical potential.

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1. Introduction

Recurrence plots (RPs) were first introduced by Eckmann et al., see [1], to visualize the recurrences of dynamical systems. Patterns in an RP are commonly used to interpret system dynamics, or to provide insight into the characteristics of observed signals. Nowadays, applications of RPs are found in numerous fields of research, in particular in physiology, biology, physics, earth sciences, engineering and economy; see for instance the extensive bibliography collected on the website [2].

Given a finite-interval continuous-time real-valued signal \( x(t) \), with \( t \in [0, 1) \) say, its corresponding recurrence plot is constructed with the time-delay embedding method (cf. [3]) in three steps:

1. An embedding dimension \( M \) and a time-delay \( \tau \in (0, 1) \) are chosen, and an associated \( M \)-dimensional vector trajectory \( X(t) \) is constructed as:

\[
X(t) = \begin{pmatrix}
x(t) \\
x(t + \tau) \\
\vdots \\
x(t + (M - 1)\tau)
\end{pmatrix}, \quad t \in [0, 1).
\]

(1)

Here, for technical convenience (to avoid having to take finite interval effects explicitly into account), the signal \( x(t) \) is periodically extended from the interval \( [0, 1) \) to all of \( \mathbb{R} \).

2. The unthresholded recurrence plot (URP) is defined as (the graph of) the intra-trajectory distance function \( \text{URP}_X(u, v) \), given by

\[
\text{URP}_X(u, v) = \| x(u) - x(v) \|, \quad u, v \in [0, 1),
\]

(2)
in which \( \| \cdot \| \) denotes the Euclidean norm.

(3) For a given positive threshold \( \epsilon \), the binary (thresholded) recurrence plot is defined as

\[
\text{RP}_X^\epsilon(u, v) = \Theta(\epsilon - \| x(u) - x(v) \|), \quad u, v \in [0, 1),
\]

in which \( \Theta \) denotes the Heaviside function (i.e. \( \Theta(x) = 1 \) for \( x \geq 0 \) and \( \Theta(x) = 0 \) otherwise).

In this Letter we shall be exclusively concerned with URPs and the information they contain. In this respect we build upon the work of [4]. The information content of binary (thresholded) recurrence plots has been studied elsewhere, see [5,6] and in particular [7].

A URP can be viewed as a transformation of a scalar signal into a 2-dimensional signal, with the \( M \)-dimensional trajectory space acting as an intermediate vehicle. In the literature, many other popular techniques exist which also produce a 2-dimensional representation of a scalar signal, such as the short-time Fourier transform (STFT) and the wavelet transform (WT). These two transforms both differ from the URP by simultaneously providing time domain (localization) information and frequency (scale) information, whereas the URP is built from time domain information only, through correlation across the entire interval. The question arises if such 2-dimensional representations still contain all the information about the underlying signal. For the STFT and the WT the answer is affirmative, as both are known to be invertible (for a suitable choice of the window and the wavelet, respectively). In the case of the WT, this property is captured by the admissibility condition, first discussed in [8].
By analyzing a recurrence plot, one intends to extract useful dynamical information from the underlying scalar signal. To quantify patterns that occur in RPs, several measures have been proposed in the literature, see e.g. [9], that are used in recurrence quantification analysis (RQA). Obviously, if it is possible for different morphologies in a signal to give rise to identical patterns in the URP or RP, then this makes it hard to attach clear meaning to these RQA measures. This motivates the central question of this Letter: to investigate the reconstructibility of a signal x(t) from its URP \|X(u) - X(v)\|, for specified values of M and \tau. Note that for M = 1 this problem is easily solved, so that we shall be concerned exclusively with M \geq 2.

In McGuire et al. [4] it was shown for discrete-time (sampled) signals that the reconstruction of a trajectory X(t) from its URP is unique up to a linear isometry. There, the trajectories are all extended with an initial zero vector, to collect absolute distance information \|X(t)\| into the URP, in addition to the relative distance information \|X(u) - X(v)\| it normally contains. In Section 2 we refine these results. We avoid such an artificial extension and we show that, in this more general setting, the URP determines the underlying trajectory up to an affine isometry. Also, by employing an arbitrary choice of time domain, we manage to capture both the discrete-time and the continuous-time case in a single framework.

When addressing the reconstruction of the underlying scalar signal x(t) which generates a trajectory X(t) with a time-delay embedding structure, the question arises for which affine isometries the time-delay embedding structure is preserved. In Section 3 we study this issue for zero mean continuous-time signals on \([0, 1]\) which admit a Fourier series representation. It is found that a URP determines the power spectrum of such x(t) for any choice of M and \tau. We also provide joint conditions on M and \tau which guarantee that a zero mean signal x(t) can be uniquely recovered from its URP up to a sign factor. When these uniqueness conditions are not satisfied, then it depends on M, \tau, and the frequency content of the actual signal x(t) itself, whether it can be recovered (up to a sign) from its URP or not. We present a constructive graph theoretical procedure to analyze this question for a given URP in full detail.

In Section 4 we propose a distance measure for trajectories in terms of their associated URPs and we investigate some of its properties. This distance measure allows us to derive the results of the previous sections in an alternative way for continuous-time signals, thus providing valuable additional insight. It also serves to address the reconstruction of a signal underlying a URP as a minimization problem. This method is of interest if one attempts to compute an approximation to an underlying signal within a parameterized class; e.g., when the URP is subject to disturbances or noise. This is covered in Section 5, where we also describe how the Fourier coefficients of a signal can be computed analytically from its URP.

Finally, in Section 6 the uniqueness conditions for URPs and their associated signals and the methods for signal reconstruction are illustrated and evaluated by two examples. First, the URPs for four different signals with coinciding power spectra are investigated for different choices of M and \tau. Second, an application with real measurement data from EEG analysis involving a Mu rhythm is presented, to illustrate the potential of our approach to approximate signals and their URPs. It also serves to demonstrate the limitations that apply to the interpretation of a URP, emanating from the choice of embedding dimension and time-delay. Section 7 concludes the Letter. All the proofs are collected in Appendix A.

2. Uniqueness of trajectories underlying an unthresholded recurrence plot

In this section we refine the results of [4] on the reconstructibility of a trajectory from its URP. To address the discrete-time case and the continuous-time case jointly in a single framework, we let \(T \subseteq \mathbb{R}\) be an arbitrary choice of time domain. For \(t \in T\), we consider two real-valued M-dimensional trajectories X(t) and Y(t). Their respective associated unthresholded recurrence plots are defined as URPx(u, v) = \|X(u) - X(v)\| and URPy(u, v) = \|Y(u) - Y(v)\|, for all u, v \in T. A URP only carries relative distance information between the points of the underlying trajectory, so that it is natural to study it with the tools from affine geometry.\(^2\)

Let \(V_X\) be the smallest affine subspace of \(\mathbb{R}^M\) which contains \(\{X(t) \mid t \in T\}\). Denote its dimension by \(k = \dim(V_X)\), and let \(\{t_0, t_1, \ldots, t_k\} \subseteq T\) be a suitably selected set of \(k + 1\) time instants such that the corresponding set \(\{X(t_0), X(t_1), \ldots, X(t_k)\}\) constitutes an affine basis of \(V_X\). Thus, every point \(X_0\) in \(V_X\) (and every point \(X(t)\) on the trajectory in particular) can be written in a unique way as a weighted linear combination of the vectors in this affine basis with weights \(\{x_{01}, \ldots, x_{0k}\}\) that add up to 1. Using the same set of time instants, a similar construction can be applied to the trajectory Y(t). The following lemma shows that any such point \(X_0\) can serve as a reference point, with respect to which the distance information between points on the trajectory X(t), as contained in the URP, is equivalently characterized through angular information, as provided by inner products denoted as \(\langle \cdot, \cdot \rangle\).

**Lemma 2.1.** Let \(\{t_0, t_1, \ldots, t_k\} \subseteq T\) be a set of time instants such that \(\{X(t_0), X(t_1), \ldots, X(t_k)\}\) is an affine basis of \(V_X\). Let \(X_0 = a_0 X(t_0) + a_1 X(t_1) + \cdots + a_k X(t_k)\) be an arbitrary point in \(V_X\), with its affine coordinates satisfying \(a_0 + a_1 + \cdots + a_k = 1\). Using the same time instants and affine coordinates, define \(Y_0 = a_0 Y(t_0) + a_1 Y(t_1) + \cdots + a_k Y(t_k)\). Then the two unthresholded recurrence plots URPx and URPy coincide if and only if for all \(u, v \in T\) it holds that \(\langle X(u) - X_0, X(v) - X_0 \rangle = \langle Y(u) - Y_0, Y(v) - Y_0 \rangle\).

This lemma allows one to show that if URPx and URPy coincide, then for all \(t \in T\) the affine coordinates of X(t) with respect to \(\{X(t_0), X(t_1), \ldots, X(t_k)\}\) are in fact identical to the affine coordinates of Y(t) with respect to \(\{Y(t_0), Y(t_1), \ldots, Y(t_k)\}\).

It is not difficult to see that whenever Y(t) is constructed from X(t) by means of an affine isometry (i.e., there is an \(M \times M\) orthogonal matrix Q and an \(M\)-vector R for which \(Y(t) = Q X(t) + R\) for all \(t \in \mathbb{R}\), then the unthresholded recurrence plots URPy and URPy coincide. (Affine isometries are commonly employed to describe rigid body motion; here the trajectories act as the rigid bodies.) The following theorem confirms that the converse statement also holds true, which establishes the main result of this section.

**Theorem 2.2.** Two unthresholded recurrence plots URX and URY coincide if and only if their underlying trajectories X(t) and Y(t) are related by an affine isometry.

This theorem refines the earlier work of [4] in three ways. First, the time domain is allowed to be an arbitrary subset of \(\mathbb{R}\). Thus, this theorem holds for trajectories and URPs in discrete-time and in continuous-time all the same. Second, it is no longer required that the trajectories are artificially initialized with a zero vector. In fact, the proof of the theorem makes clear that the translated trajectories \(\tilde{X}(t) = X(t) - X_0\) and \(\tilde{Y}(t) = Y(t) - Y_0\) are related by a linear isometry for arbitrary \(X_0 \in V_X\) and a matching choice of \(Y_0\). If \(X_0 = X(t_0) = 0\) and \(Y_0 = Y(t_0) = 0\) are chosen, then the result of [4] is obtained as a special case with \(t_0 = 0\). In

\(^2\) In affine geometry there is no preferred choice of the origin, so that a basis for an n-dimensional (affine) subspace is defined in terms of \(n + 1\) points in the space, rather than by \(n\) vectors spanning it.
Sections 3 and 4 though, when dealing with continuous-time signals, it will be convenient instead to choose \( X_0 = \int_0^1 X(t) \, dt \) and \( Y_0 = \int_0^1 Y(t) \, dt \). Third, in [4] the proof requires that the dimension \( k \) of the affine subspace \( \mathcal{V} \) is equal to the dimension \( M \) of the embedding space; the degenerate case \( k < M \) is left undressed. Here it is found that the theorem applies to this case too. Then the orthogonal matrix \( Q \) and the associated isometry are no longer unique.

If one attempts to reconstruct a trajectory from its URP, then the constructive proof of Lemma 2.1 provides a method to achieve this. Note that the affine coordinates of any point \( x(t) \) with respect to the selected affine basis \( \{X(t_0), X(t_1), \ldots, X(t_p)\} \) can be computed directly from the data in the URP \( \|X(u) - X(v)\|_2 \), as indicated there. These coordinates allow for a natural representation of \( X(t) \) in \( \mathbb{R}^k \); the corresponding inner product involves a Gram matrix which is directly computable from the URP. Then it is straightforward to compute a linear transformation which allows one to represent the trajectory \( X(t) \) in \( k \)-dimensional Euclidean space; e.g., by computing a square root of the Gram matrix.

3. Uniqueness of scalar signals underlying an unthresholded recurrence plot generated with the time-delay embedding method

We now turn to the issue of reconstructing a scalar signal \( x(t) \) from a URP. The result of the previous section establishes that a URP determines the underlying trajectory up to an affine isometry. When the time-delay embedding method is used to generate the trajectory \( X(t) \), this naturally raises the question to which extent the time-delay embedding structure of \( X(t) \) can remain preserved under affine isometric transformation. This depends on the choice of time-delay \( \tau \), embedding dimension \( M \), and on the signal \( x(t) \) itself.

Motivated by practical applications, we focus on real-valued continuous-time signals on a finite interval. To handle the finite interval effects conveniently (the time-delay embedding procedure employs shifted signals!), we extend all signals by making them integrable on \( [0, 1] \). Note that the affine coordinates of any point \( x(t) \) denote the mean value of the trajectory \( X(t) \) attains the constant value

\[
X_0 = \frac{1}{0} X(t) \, dt = c_0 \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right).
\]

It constitutes a point in the affine space \( \mathcal{V} \) which provides a convenient reference point for studying the trajectory \( X(t) \). Evidently, the URP does not contain any information on \( c_0 \). Therefore, without loss of generality, we further restrict the class of signals \( x(t) \) to those of zero mean, by setting \( c_0 = 0 \).

For such a zero mean scalar signal \( x(t) \), the trajectory \( X(t) \) is also periodic with period 1. It is given by:

\[
X(t) = \sum_{k \in \mathbb{Z}} c_k e^{2 \pi i k t} \, T_k,
\]

where \( M \) is the dimension of the embedding space; e.g., by computing a square root of the Gram matrix.

in which (for all \( k \in \mathbb{Z} \))

\[
T_k = \left( \begin{array}{c} 1 \\ z^k \\ \vdots \\ z^{(M-1)k} \end{array} \right), \quad \text{with} \quad z = e^{2 \pi i \tau}.
\]

When considering the inner product \( \langle X(u), X(v) \rangle \) as a two-variable function of \( u, v \in [0, 1] \), it is obtained that

\[
\langle X(u), X(v) \rangle = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} c_p c_q e^{2 \pi i (pu - qv)} \langle T_p, T_q \rangle.
\]

This constitutes a 2-dimensional Fourier series representation of \( \langle X(u), X(v) \rangle \), with 2D-Fourier coefficients \( c_p c_q \). Such a representation is unique, and in view of Lemma 2.1 with \( X_0 = 0 \), we therefore have that specification of these 2D-Fourier coefficients is equivalent to specification of the URP. The following lemma provides an explicit expression for the inner product \( \langle T_p, T_q \rangle \).

**Lemma 3.1.** For the sesquilinear inner product \( \langle T_p, T_q \rangle \), with linearity in its first argument, it holds that:

\[
\langle T_p, T_q \rangle = \sum_{m=1}^M \sum_{n=1}^M (z^{p-q}(m-1))
\]

\[
= \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} c_p c_q e^{2 \pi i (pu - qv)} \langle T_p, T_q \rangle.
\]

Consequently, \( \langle T_p, T_q \rangle \neq 0 \) if and only if \( \langle T_p, T_q \rangle \neq 0 \) for \( (p - q) \tau \in \mathbb{Z} \), \( \langle T_p, T_q \rangle \neq 0 \) for \( (p - q) \tau \notin \mathbb{Z} \).

For \( p = q \) it follows that \( \langle T_p, T_q \rangle = M \neq 0 \), so that the corresponding 2D-Fourier coefficient attains the value \( M ||p||^2 \). The following theorem now follows, which states that a URP totally determines the power spectrum of its underlying signal.

**Theorem 3.2.** Let \( \tau \in (0, 1) \) and \( M \geq 1 \) be given and let \( x(t) \) and \( y(t) \) be two real-valued scalar continuous-time zero-mean periodic signals of period \( T = 1 \), which are square integrable on \( (0, 1) \). If the unthresholded recurrence plots URP\(_X \) and URP\(_Y \) coincide, then also the power spectra of \( x(t) \) and \( y(t) \) coincide.

To answer the question to which extent knowledge of the 2D-Fourier coefficients \( c_p c_q \) allows one to reconstruct the signal \( x(t) \), we address the computation of the Fourier coefficients \( c_k \) from these quantities. Note that Theorem 3.2 makes clear that the URP allows one to compute which coefficients \( c_k \) are zero and which are not; for any nonzero coefficient it fixes its modulus.

If two signals with coefficient series \( \{c_k\} \) and \( \{d_k\} \), respectively, yield identical URPs, then for all \( k \) such that \( c_k \neq 0 \) there exists a unique unimodular constant \( \gamma_k \) such that \( d_k = \gamma_k c_k \). If \( p \) and \( q \) are two indices with \( c_p \neq 0 \), \( c_q \neq 0 \), and \( \langle T_p, T_q \rangle = 0 \), then \( c_p c_q \langle T_p, T_q \rangle = d_p d_q \langle T_p, T_q \rangle \) implies \( \gamma_p = \gamma_q \). This motivates the following graph theoretic construction.

**Definition 3.3.** For a given unthresholded recurrence plot URP\(_X \) generated with the time-delay embedding method from a zero mean scalar signal \( x(t) = \sum_{k \in \mathbb{Z}} c_k e^{2 \pi i k t} \), let \( K_X \subset \mathbb{Z} \setminus \{0\} \) be the set of indices for which \( c_k \neq 0 \). Define the associated undirected simple graph \( G_X \) as follows:

1. its nodes are uniquely labeled by the numbers \( k \in K_X \);
2. two nodes labeled \( p \) and \( q \in K_X \) are adjacent if and only if \( \langle T_p, T_q \rangle \neq 0 \).

Reconstructibility of a signal \( x(t) \) from its URP can now be characterized, constructively, in terms of the graph \( G_X \).
Theorem 3.4. Let \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \) and \( y(t) = \sum_{k=0}^{K} d_k e^{2\pi i k t} \) be two Fourier representations of real-valued zero mean scalar signals (with period \( T = 1 \), \( c_0 = d_0 = 0 \), and square integrable on \([0, 1]\)). If the unthresholded recurrence plots \( URP_X \) and \( URP_Y \) coincide, then the graph \( G_X \) is connected, then the signal \( x(t) \) is determined up to a sign choice, i.e., \( y(t) = \pm x(t) \) for all \( t \in [0, 1] \).

Conversely, if the graph \( G_X \) is not connected, then there exist different such signals \( x(t) \) and \( y(t) \) with identical unthresholded recurrence plots \( URP_X \) and \( URP_Y \).

Reconstructibility involves the embedding dimension \( M \) and the time-delay \( \tau \), which jointly determine the pairs \((p, q)\) for which \( t(T_p, T_q) \neq 0 \), generating edges in the graph \( G_X \). It also involves the signal \( x(t) \), in particular its frequency content, as only the non-zero coefficients \( c_k \) generate nodes in the graph \( G_X \). Below, a number of special cases are addressed for which connectedness of the graph \( G_X \) can be established directly.

Corollary 3.5.

(1) Let \( \tau \not\in \mathbb{Q} \). Then \( G_X \) is a complete graph, hence connected.

(2) Let \( \tau \in \mathbb{Q} \) and let \( d \) be the denominator of \( \tau \). If \( M \) and \( d \) are co-prime, then \( G_X \) is a complete graph, hence connected.

(3) Let \( \tau \in \mathbb{Q} \). Suppose that \( x(t) \) is a zero mean real signal which has a finite Fourier series representation \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \). If the denominator of \( M \tau \) exceeds \( 2K \), then \( G_X \) is a complete graph, hence connected.

(4) Let \( M t \in \mathbb{Z} \). If two nodes \( p, q \in K_X \) satisfy \((p - q) \tau \not\in \mathbb{Z}\), then \( G_X \) is disconnected, and \( p \) and \( q \) belong to different connected components of \( G_X \).

(5) Let \( G_X \) be disconnected, so that \( \tau \in \mathbb{Q} \), and let \( d \) be the denominator of \( \tau \). Then each connected component of \( G_X \) constitutes a complete subgraph of \( G_X \). Any two distinct nodes \( p, q \in K_X \) are adjacent if and only if \((p - q) \tau \in \mathbb{Z} \). The number of connected components of \( G_X \) is finite: it equals the number of nonempty equivalence classes of \( K_X \) modulo \( d \).

(6) Let \( \tau \in \mathbb{Q} \) and let \( d \) be the denominator of \( \tau \). Suppose that for two nodes \( p, q \in K_X \) it holds that \((p - q) \tau \not\in \mathbb{Z} \). Then \( G_X \) is disconnected if and only if \( M t \in \mathbb{Z} \), if and only if \( p \) and \( q \) are not adjacent, if and only if \( p \) and \( q \) belong to different connected components of \( G_X \).

(7) Let \( M t \not\in \mathbb{Z} \) and let \( d \) be the denominator of \( \tau \). Suppose that \( x(t) \) is a zero mean real signal which has a finite Fourier series representation \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \). If \( d > 2K \) then \( G_X \) is totally disconnected.

Parts (5) and (6) of Corollary 3.5 offer powerful tools to quickly determine connectedness of \( G_X \). If \( p, q \in K_X \) are adjacent and belong to different equivalence classes modulo the denominator \( d \) of \( \tau \), then \( G_X \) is connected. If \( p \) and \( q \) belong to the same equivalence class modulo \( d \) while not being adjacent, \( G_X \) is also connected. And often it will not be hard to select \( p \) and \( q \) from \( K_X \) such that \( p - q \) and \( d \) are co-prime.

The statement of Theorem 3.4 can be refined by not just considering connectedness of \( G_X \), but taking all its connected components into account. If \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \) and \( y(t) = \sum_{k=0}^{K} d_k e^{2\pi i k t} \) have the same URP, then each connected component of \( G_X \) links all its associated coefficients \( c_k \) to \( d_k \) by means of a single unimodular coefficient \( \gamma_k \) (i.e., which is constant on that connected component). When \( x(t) \) and \( y(t) \) are real, the coefficients \( \gamma_k \) additionally satisfy the relationships \( \gamma_{-k} = \gamma_k \). We then have the following result.

Corollary 3.6.

(1) If \( M t \in \mathbb{Z} \) then the signals \( \pm x(t + \tau t) \) \((\tau \in \mathbb{Z})\) all have identical URPs.

(2) Let \( x(t) \) and \( y(t) \) be two zero mean real signals from our class. If \( G_X \) is totally disconnected, then \( URP_X = URP_Y \) if and only if \( x(t) \) and \( y(t) \) have the same power spectrum.

(3) Let \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \) and \( y(t) = \sum_{k=0}^{K} d_k e^{2\pi i k t} \) be two zero mean real signals having finite Fourier series representations. Then \( URP_X = URP_Y \) if and only if \( y(t) = (x \pm y)(t) := \int_0^T x(s) \gamma(t - s) ds \) is the convolution of \( x(t) \) and some zero mean real function \( \gamma(t) = \sum_{k=0}^{K} \gamma_k e^{2\pi i k t} \) such that \( |\gamma_k| = 1 \) for all \( k \in K_X \), and \( \gamma_p = \gamma_q \) if \( c_p \gamma_q(T_p, T_q) \neq 0 \).

In the situation of part (2) of Corollary 3.6 it follows that: \((X(u), X(v)) = M \sum_{k=0}^{K} c_k^2 e^{2\pi i k(u-v)} \), whence \( |X(u) - X(v)|^2 = 2M \sum_{k=0}^{K} |c_k|^2 (1 - e^{2\pi i k(u-v)}) \), which depends only on the difference \( u - v \). Therefore, the URP of a signal for which \( G_X \) is totally disconnected is banding, constant along diagonals on which \( u - v \) remains constant.

Also note that part (3) of Corollary 3.6 provides a time-domain characterization of coinciding URPs in terms of convolution. However, it only applies to signals having finite Fourier series representations, because \( \gamma(t) \), in view of unimodularity of the coefficients \( \gamma_k \), has finite power if and only if \( K_X \) is finite.

In a discrete-time setting, one typically has to deal with a finite data record \( X[0], X[1], \ldots, X[N-1] \) of \( N \) sampled real values at equally spaced time instants. Then the sampling time can be chosen as \( 1/N \) and an interpolating periodic signal \( x(t) \) of period \( T = 1 \) can be uniquely constructed as a finite Fourier series: \( x(t) = \sum_{k=0}^{K} c_k e^{2\pi i k t} \) with \( c_k = c_k \) for all \( k \) and satisfying \( x(t/N) = x(t) \) for all \( t \in [0, 0, \ldots, N-1] \). Here \( K = \lfloor N/2 \rfloor \) so that \( 2K = N \) if \( N \) is even, and \( 2K = N-1 \) if \( N \) is odd; in the former case one additionally requires that \( c_K = c_{-K} \) is real.

It is natural only to consider values of \( \tau \) with denominators \( d \) that are factors of \( N \), as it allows the trajectory \( X(t) \) to be constructed from \( x(t) \) by shifting the measured signal, without interpolation. From part (2) of Corollary 3.6 we now have that if \( M \) and \( d \) are co-prime, then the graph \( G_X \) is connected. Consequently, then the URP computed at all pairs of sample points \((u, v)\) only, determines the de-averaged signal \( x(t) = c_0 \) up to a sign \( \pm 1 \).

4. A distance measure for unthresholded recurrence plots

When comparing different URPs, it is convenient to have a measure which quantifies their difference. In this section we introduce and study such a measure, which turns out to be useful in two respects. First, it allows us to recover Lemma 2.1 in an alternative way. Second, it facilitates the approximation of a noisy signal (underlying a noisy URP) by a signal from a parameterized class, by minimizing the difference between their URPs (see Section 5). We have the following definition.

Definition 4.1. Let \( X(t) \) and \( Y(t) \), with \( t \in [0, 1] \), be two continuous-time trajectories in \( M \)-dimensional space, with associated unthresholded recurrence plots \( URP_X \) and \( URP_Y \). A distance measure \( F \) and a related measure \( \tilde{F} \) which quantify the difference between \( X(t) \) and \( Y(t) \) through the difference between their squared URPs, are defined as:

\[
F(X, Y) := \tilde{F}(URP_X, URP_Y) := \int_0^1 \int_0^1 (URP_X(u, v) - URP_Y(u, v))^2 \, du \, dv.
\]
**Proposition 4.2.** For $F(X, Y)$ and $\mathcal{F}(\text{URP}_X, \text{URP}_Y)$ introduced above, it holds that:

1. $F(X, Y) = \mathcal{F}(\text{URP}_X, \text{URP}_Y) \geq 0$;
2. $F(X, Y) = \mathcal{F}(\text{URP}_X, \text{URP}_Y) = 0$ if and only if $\text{URP}_X = \text{URP}_Y$;
3. $F(X, Y) = F(Y, X)$ and $\mathcal{F}(\text{URP}_X, \text{URP}_Y) = \mathcal{F}(\text{URP}_Y, \text{URP}_X)$;
4. $F(X + X_0, Y + Y_0) = F(X, Y)$ for all $X_0, Y_0 \in \mathbb{R}^M$, and $\text{URP}_X + X_0 = \text{URP}_X$ for all $X_0 \in \mathbb{R}^M$.

For zero mean trajectories $X(t)$ and $Y(t)$, the distance measure $F(X, Y)$ admits an interesting decomposition into nonnegative terms, as put forward in the following proposition.

**Proposition 4.3.** If $X_0 := \int_0^1 X(t) \, dt = 0$ and $Y_0 := \int_0^1 Y(t) \, dt = 0$, then $F(X, Y)$ can be written as a sum of three nonnegative terms as:

$$F(X, Y) = 2N_1(X, Y) + 2N_2(X, Y) + 4I(X, Y),$$

where

$$N_1(X, Y) = \int_0^1 \left( \|X(t)\|^2 - \|Y(t)\|^2 \right) \, dt,$$

$$N_2(X, Y) = \left( \int_0^1 \left( \|X(t)\|^2 - \|Y(t)\|^2 \right) \, dt \right)^2,$$

$$I(X, Y) = \int_0^1 \int_0^1 \left( \langle X(u), X(v) \rangle - \langle Y(u), Y(v) \rangle \right)^2 \, du \, dv.$$

Note also that, as a special case of Eq. (13) with $u = v = t$, one can compute $\|X(t)\|^2$ as:

$$\|X(t)\|^2 = \int_0^1 \text{URP}_X(t, \tilde{v}) \, d\tilde{v} - \frac{1}{2} \int_0^1 \text{URP}_X(\tilde{u}, \tilde{v}) \, d\tilde{u} \, d\tilde{v}. \quad (15)$$

The latter expression makes it possible to reconstruct a trajectory $X(t)$ alternatively, in a point by point fashion, in the following way. First, a time instant $t_1$ is selected and $\|X(t_1)\|$ is computed. Then the point $X(t_1)$ is reconstructed accordingly in $\mathbb{R}^M$, to be located in the direction of the positive first coordinate axis. Next, a second time instant $t_2$ is selected and the distances $\|X(t_1)\|$ and $\|X(t_1) - X(t_2)\| = \text{URP}_X(t_1, t_2)$ are computed. This determines a unique location for $X(t_2)$ in the half-plane spanned by the first coordinate axis and the positive second coordinate axis. We proceed with time instants $t_3, t_4, \ldots$ in a similar fashion, comparing with at most $M$ previously computed points, employing new coordinate axes only when necessary.

When a trajectory is generated by the time-delay embedding method, then to achieve the reconstruction of the underlying zero mean scalar signal $x(t)$ from its URP, one may equivalently proceed to reconstruct its Fourier coefficients. The following proposition addresses this, in terms of $(X(u), X(v))$. As we have seen, it depends on connectedness of the associated graph $G_X$ whether $x(t)$ can be reconstructed up to a sign $\pm 1$.

**Proposition 5.2.** Let $x(t) = \sum_{k \in \mathbb{Z}} c_k \, e^{2\pi i k t}$ with $c_0 = 0$ be a (scalar) zero mean real signal, which, for a given choice of embedding dimension $M$ and time-delay $\tau$, generates a zero mean trajectory $X(t)$ and a given unthresholded recurrence plot $\text{URP}_X$. Then:

1. The squared modulus of a Fourier coefficient $c_k$ is obtained as:

$$P(k) := |c_k|^2 = \int_0^1 \int_0^1 \langle X(u), X(v) \rangle \cos(2\pi k(u - v)) \, du \, dv. \quad (16)$$

2. If two nodes $p$ and $q$ in the graph $G_X$ are adjacent, then the quotient of $c_p$ and $c_q$ satisfies:

$$\frac{c_p}{c_q} = \int_0^1 \int_0^1 \langle X(u), X(v) \rangle e^{-2\pi (pu - qv)} \, du \, dv. \quad (17)$$

3. If two nodes $k$ and $-k$ in the graph $G_X$ are connected by a path $k = n_0 \leftrightarrow n_1 \leftrightarrow \cdots \leftrightarrow n_{\ell-1} \leftrightarrow n_\ell = -k$, then:

$$c_k^2 = P(k) \prod_{i=1}^{\ell} Q(n_{i-1}, n_i). \quad (18)$$

For a signal $x(t)$ with a connected graph $G_X$ we have that: part (1) of this proposition allows one to determine which Fourier coefficients are nonzero, part (3) allows one to determine the value of a selected nonzero coefficient $c_k$ up to a sign, and part (2) allows one to compute the values of all the other nonzero coefficients by following paths which link them to $c_k$. Should $G_X$ have a more than one connected component, then this proposition allows one to parameterize the class of all signals $x(t)$ that are consistent with the given URP.
As we have seen in Section 3, Eq. (6), the quantities \( c_p^T(T_a, T_b) \) show up as the 2D-Fourier coefficients of the two-variable function \( \langle X(u), X(v) \rangle \), which can all be computed from \( URP_X \). It follows that one can decide from a given \( URP \), without explicit knowledge of \( \tau \) or \( M \), whether the underlying signal \( x(t) \) can be reconstructed up to a sign or not. To actually reconstruct \( x(t) \) one will need \( \tau \) and \( M \). When \( G_X \) is connected then the expressions of Lemma 3.1 help to constrain any feasible values of \( \tau \) and \( M \).

An alternative approach to the reconstruction of a zero mean signal \( x(t) \) from a given unthresholded recurrence plot \( URP_0 \), is to minimize the distance measure \( \tilde{F}(URP_X, URP_0) \). Here, any approximating unthresholded recurrence plot \( URP_X \) is regarded to be defined in terms of \( x(t) \), which may be taken from a suitable parameterized class. Such an approach is particularly attractive in noisy situations, or when the given unthresholded recurrence plot \( URP_0 \) is computed from a sampled signal. As a consequence of Parseval’s identity, the computation of the distance measure \( \tilde{F}(URP_X, URP_0) \) is easily performed in the frequency domain.

**Proposition 5.3.** Let two squared unthresholded recurrence plots \( URP_X^2 \) and \( URP_Y^2 \) be expressed as 2-dimensional Fourier series:

\[
URP_X(u, v)^2 = \sum_{p \in \mathbb{Z}, q \in \mathbb{Z}} C_{pq} e^{2\pi i (pu + qv)},
\]

\[
URP_Y(u, v)^2 = \sum_{p \in \mathbb{Z}, q \in \mathbb{Z}} D_{pq} e^{2\pi i (pu + qv)},
\]

Then the distance measure \( \tilde{F}(URP_X, URP_Y) \) is given by:

\[
\tilde{F}(URP_X, URP_Y) = \sum_{p \in \mathbb{Z}, q \in \mathbb{Z}} |C_{pq} - D_{pq}|^2.
\]  

(19)

When \( x(t) \) is modeled to contain just a few dominant frequencies, then a good initial point for the numerical minimization of \( \tilde{F}(URP_X, URP_0) \) is obtained by restricting \( URP_X \) to the same frequencies in both dimensions.

6. Examples and an application in EEG analysis

To illustrate the main results and techniques of the previous sections, we here present two examples. These examples serve to demonstrate the limitations that apply to the interpretation of a \( URP \), emanating from the choice of embedding dimension and time-delay.

In the first example we consider four different signals which share the same power spectrum. We investigate the impact of different choices for \( M \) and \( \tau \), which may cause morphologically different signals to exhibit identical \( URP \).

In the second example we investigate an application in EEG analysis, which concerns a digitally sampled measurement signal featuring a so-called Mu rhythm. The distance measure \( F(X, Y) \) is employed to construct an approximation of the measured Mu rhythm, by minimization over a parameterized class of signals, which is generated in accordance with the dominant frequencies of the Mu rhythm. Again, the selection of \( M \) and \( \tau \) is shown to have impact on the information content of the \( URP \).

**Example 1.** Different signals exhibiting identical \( URP \).

We consider four different zero mean real signals (see Fig. 1) with identical power spectra, involving just two different frequencies.

\[
a(t) = 2 \sin(2\pi t) - \sin(4\pi t),
\]

(20)

\[
b(t) = -2 \cos(2\pi t) - \cos(4\pi t),
\]

(21)

\[
c(t) = 2 \cos \left( 2\pi t + \frac{\pi}{8} \right) + \cos(4\pi t),
\]

(22)

\[
d(t) = 2 \cos(2\pi t + \frac{\pi}{8}) - \cos(4\pi t).
\]

(23)

Note, that the signals \( a(t) \), \( b(t) \), and \( c(t) \) are all morphologically different, whereas \( d(t) \) is a time-shifted version of the signal \(-c(t)\): it holds that \( d(t) = -c(t + \frac{1}{2}) \). Writing \( a(t) = \sum_{k=-2}^{2} a_k e^{2\pi k i t} \), \( b(t) = \sum_{k=-2}^{2} b_k e^{2\pi k i t} \), and so on, the Fourier coefficients are given by:

\[
a(t): \quad a_1 = \bar{a}_{-1} = -i, \quad a_2 = \bar{a}_{-2} = \frac{1}{2} i;
\]

\[
b(t): \quad b_1 = \bar{b}_{-1} = -1, \quad b_2 = \bar{b}_{-2} = -\frac{1}{2} i;
\]

\[
c(t): \quad c_1 = \bar{c}_{-1} = e^{\frac{\pi i}{4}}, \quad c_2 = \bar{c}_{-2} = \frac{1}{2} i;
\]

\[
d(t): \quad d_1 = \bar{d}_{-1} = e^{\frac{\pi i}{4}}, \quad d_2 = \bar{d}_{-2} = -\frac{1}{2} i.
\]
It is noted that for each signal: \( P(1) = P(-1) = 1 \) and \( P(2) = P(-2) = \frac{3}{4} \), see Eq. (16), which confirms that the power spectra of the signals are indeed identical.

The associated graphs \( \mathcal{G}_A, \mathcal{G}_B, \mathcal{G}_C \) and \( \mathcal{G}_D \) all have exactly 4 nodes, labeled \(-2, -1, 1\) and 2. Adjacency of those nodes depends on the values of \( M \) and \( \tau \). An edge exists between two nodes \( p \) and \( q \), if and only if either \((p - q) \tau \in \mathbb{Z}\) or \((p - q) M \tau \notin \mathbb{Z}\).

Note that, for the given signals, \( p - q \) can attain all integer values in the range between \(-4\) and \(4\).

Here we investigate the URPs of these four signals for five different settings of \( M \) and \( \tau \):

(i) \( M = 6 \) and \( \tau = \frac{1}{6} \). In view of part (7) of Corollary 3.5 it holds that \( \tau = \frac{1}{M} \) and \( M > 2K \). Therefore the graph is totally disconnected so that all signals with identical power spectra have
are adjacent, whence connectedness of the graph holds accord-

\[ \gamma(\tau t) := 2\cos(2\pi t + \frac{1}{4}) - 2\cos(4\pi t), \]

for which \( \gamma(t) = (b \ast \gamma'(t)) \). The URPs of the morphologically different signals \( b(t) \) and \( c(t) \) coincide.

Similarly, the signal \( d(t) \) is related to the signal \( c(t) \) by \( d_k := c_k \gamma_k \), where the unimodular constants \( \gamma_k \) are now given by \( \gamma_1 = \gamma_{-1} = 1 \) and \( \gamma_2 = \gamma_{-2} = -1 \). Again, note that \( (T_p, T_q) \neq 0 \) and \( \gamma_p = \gamma_q \) for the node pair \( (p, q) = (2, 2) \). In this case, the signal \( \gamma'(t) \) is given by \( \gamma'(t) := 2\cos(2\pi t) - 2\cos(4\pi t) \), for which \( d(t) = (c \ast \gamma'(t)) \). Therefore, the URPs of the signals \( c(t) \) and \( d(t) \) coincide too. Alternatively, this is confirmed by part (1) of Corollary 3.6, since \( M_0 = \Sigma \) and \( d(t) = -c(t + \frac{1}{2}) = -c(t + 2\tau) \). Summa-

\[ \gamma_{18}2 \ast \gamma_{16} \]

rizing: the URPs \( b(t) \), \( c(t) \) and \( d(t) \) are all identical, the URP of \( a(t) \) is different.

(iii) \( M = 4 \) and \( \tau = \frac{1}{2} \). The graph has two different connected components, see Fig. 2(iii). As in case (ii), the signal \( d(t) \) can be constructed from the signal \( c(t) \), where in the present case \( d(t) = -c(t + \frac{1}{2}) = -c(t + \tau) \). Therefore, the URPs of \( c(t) \) and \( d(t) \) again coincide, which can also be established directly from the structure of the graph and the values of the Fourier coefficients of the signals. The URPs of the signals \( a(t) \) and \( b(t) \) are different.

(iv) \( M = 6 \) and \( \tau = \frac{1}{2} \). The graph has two different connected components, see Fig. 2(iv). In this case the signal \( a(t) \) is related to the signal \( b(t) \) by \( a_k := b_k \gamma_k \), where the unimodular constants \( \gamma_k \) are given by \( \gamma_1 = \gamma_2 = \gamma_{-1} = \gamma_{-2} = 1 \). Note that \( (T_p, T_q) \neq 0 \) and \( \gamma_p = \gamma_q \) for the node pairs \( (p, q) \in \{(-1, 2), (-2, 1)\} \). The associated signal \( \gamma'(t) \) is given by \( \gamma'(t) := -2\sin(2\pi t) + 2\sin(4\pi t) \), for which \( a(t) = (0 \ast \gamma'(t)) \). Therefore, the URPs of the morphologically different signals \( a(t) \) and \( b(t) \) coincide. The other two signals have different URPs.

(v) \( M = 4 \) and \( \tau = \frac{1}{3} \). In this case the graph clearly is connected, see Fig. 2. A quick way to establish connectedness, is to consider the node pair \( (p, q) = (1, 2) \), for which \( p - q = 1 \) and the denominator \( d = 6 \) of \( \tau \) are co-prime. It holds that \( (p - q)M \in \mathbb{Z} \) so that \( p \) and \( q \) are adjacent, whence connectedness of the graph holds according to part (6) of Corollary 3.5. The URP therefore determines all the Fourier coefficients of a signal jointly up to a single sign. Consequ-

\[ \gamma_{18}2 \ast \gamma_{16} \]

ently, the URPs of the signals \( a(t) \), \( b(t) \), \( c(t) \) and \( d(t) \) are all different.

For the nodes \(-1, -2, 1, 2\), other choices of \( M \) and \( \tau \) may still generate different graphs, not yet encountered in this example. These remaining graphs are displayed in Fig. 3; they are all connected. Consequently, for corresponding settings of \( M \) and \( \tau \), the URPs of the signals \( a(t) \), \( b(t) \), \( c(t) \) and \( d(t) \) are all different.

In the literature, a common approach to estimate the time-

delay \( \tau \) and the embedding dimension \( M \), employs average mutual information (AMI) [10] and the false nearest neighbors fraction (FNNF) [11], respectively. This approach, however, may lead to values of \( M \) and \( \tau \) that correspond to a disconnected graph. The following example illustrates such a situation. Consider the zero mean signal \( f(t) = 4\sin(2\pi t) + \cos(4\pi t) \), involving just two different frequencies, see Fig. 4(i). First, the AMI is computed, see Fig. 4(ii), and its first local minimum is determined: it is located approximately at \( \tau = 0.25 \). Second, \( M \) is chosen to correspond to a zero (or a sufficiently small value) of the FNNF. For \( \tau = 0.25 \), the FNNF is zero for all \( M \geq 2 \), see Fig. 4(iii). For \( M = 2 \) and \( M = 3 \) it can be verified that the associated graphs are connected: they coincide with the third and fourth graph in Fig. 3, respectively. For \( M = 4 \), however, the associated graph is disconnected: it coincides with the graph in Fig. 2(ii). Hence, for \( M = 4 \) there exist signals that are morphologically different from \( f(t) \) but which share the same URP. Consequently, for \( M = 4 \) not all the information in \( f(t) \) is stored in its URP. The software used for this example is available on the website [12].

**Example 2. EEG analysis featuring a Mu rhythm.**

From a digitally sampled EEG measurement signal we consider an excerpt of \( N = 100 \) samples, exhibiting a Mu rhythm with a duration of \( T = 0.4 \) s. Mu rhythms are important EEG rhythms which have been well studied in the literature, see [13]. They are character-

\[ \gamma_{18}2 \ast \gamma_{16} \]

ized by an \( m \)-shaped morphology and a frequency content between 8 Hz and 12 Hz. For the available sampled signal we reconstruct a continuous-time signal \( x(t) \) having a finite Fourier series, see Fig. 5(x), which interpolates the sampled measurement values.

The unthresholded recurrence plot URPX of the signal \( x(t) \) for the settings \( M = 3 \) and \( \tau = \frac{3}{20}T = 0.024 \) s is displayed in Fig. 6(x). Note that \( \tau \) is an integer multiple of the sampling time \( \Delta t := \frac{T}{4} = 0.004 \) s. The associated graph Gx has 100 nodes: \( k \in Kx = \{-50, \ldots, -1\} \cup \{1, \ldots, 50\} \).

First, we have approximated \( x(t) \) by a signal \( y(t) \), see Fig. 5(y), which has a finite Fourier series with frequencies \( f_k = \frac{k}{T} \) for \( k \in \{4, 8, 12, 16\} \). They correspond to the dominant frequencies in the power spectrum of \( x(t) \). Hence, the associated graph Gy has 8 nodes: \( k \in Ky = \{-16, -12, -8, -4, 4, 8, 12, 16\} \). To compute the corresponding 8 Fourier coefficients \( y_k \) of the approximation \( y(t) \), we have employed the minimization approach of Section 5, using a general purpose numerical optimization routine from Matlab to minimize \( F(X, Y) \).

Second, to demonstrate some pitfalls in drawing conclusions from URPs, we have constructed a real signal \( z(t) \) by altering all the phases of the Fourier coefficients \( y_k \) with positive index \( k \), by \( -\pi/2 \). I.e., the Fourier coefficients \( z_k \) of \( z(t) \) relate to those of \( y(t) \) by \( z_k = y_k / i \) for \( k = 4, 8, 12, 16 \). The resulting signal \( z(t) \)
Fig. 4. (i) Signal \( f(t) \), (ii) its average mutual information \( AMI(τ) \), (iii) and its false nearest neighbors fraction \( FNNF(M) \) for \( τ = 0.25 \).

Fig. 5. The interpolated Mu rhythm signal \( (x) \), its approximation \( (y) \), and its phase-altered approximation \( (z) \).

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has a sawtooth morphology and is displayed in Fig. 5(z). The signals \( y(t) \) and \( z(t) \) by construction have identical power spectra, but they clearly exhibit different morphologies. For the purpose of valid EEG analysis, the URPs of \( y(t) \) and \( z(t) \) should be properly distinguishable—this depends on the values of \( M \) and \( τ \).

To apply the propositions of Section 3 on the connectedness of the graphs associated with the signals \( x(t) \), \( y(t) \) and \( z(t) \), we must replace \( τ \) by the normalized time-delay \( \frac{τ}{T} \) in these propositions. Since \( M \) and the denominator of \( \frac{τ}{T} \) are co-prime, it now follows from part (2) of Corollary 3.5 that the graphs \( G_X \), \( G_Y \) and \( G_Z \) are all connected. Consequently, the URPs of the signals \( x(t) \), \( y(t) \) and \( z(t) \) are all different, as demonstrated in Fig. 6. As a consequence of the approximation approach, the URPs of the signals \( x(t) \) and \( y(t) \) are clearly more alike than the URPs of \( x(t) \) and \( z(t) \), which is desirable for making valid morphology related inferences from URP\(_X\) or its approximation URP\(_Y\).

Next, we have considered the alternative settings \( M = 25 \) and \( τ = \frac{3}{2} T = 0.024 \) s. In this extreme case, \( ⟨T_p, T_q⟩ = 0 \) for all pairs of different nodes \( p, q ∈ K_Y = K_Z = \{−16, −12, −8, −4, 4, 8, 12, 16\} \). It follows that the graphs \( G_Y \) and \( G_Z \) are now totally disconnected. Consequently, the URPs of the signals \( y(t) \) and \( z(t) \) coincide as they have identical power spectra. Their URPs are again banded, as remarked after Corollary 3.6; see Fig. 7. In this case, it is hazardous to make valid morphology related inferences from URP\(_X\) or URP\(_Y\).

In practice, signal processing is often performed by applying filters, which typically tend to reduce the frequency content of a signal \( x(t) \) to a relevant passband. If a filtered signal contains just a few dominant frequencies and is well approximated by a finite Fourier series, then as we have seen, depending on \( M \) and \( τ \), it may happen that the URP of the approximation no longer contains full information on that signal. If two continuous signals are close,
then clearly also their URPs are close. This means that even if the URP of the filtered signal theoretically allows for the reconstruction of that signal up to a sign factor, such computations are likely to become numerically ill-conditioned. It therefore is advisable to select \( M \) and \( \tau \) such that reconstruction up to a sign factor of the dominant Fourier coefficients is guaranteed.

These examples show that one may not simply assume that a URP contains all information about the signal. Essential information on the morphology of the signal can be lost by an ill choice of the embedding dimension \( M \) and time-delay \( \tau \), which is clearly to be avoided in certain applications such as EEG analysis. On the other hand, appropriate choices for \( M \) and \( \tau \) which allow signals with the same power spectra but with different morphologies to be distinguished through their URPs, are generally possible, and the results in this Letter provide means to determine such choices.

**7. Conclusions and discussion**

When considering URPs as a tool to extract information from signals, we have argued in this Letter that it is important to first establish which information can or cannot be recovered from URPs. In Section 2 it is shown that the trajectory \( X(t) \) in \( M \)-dimensional space from which a unthresholded recurrence plot URP\(_X\) is generated, can always be recovered up to an affine isometry (refining related results in the literature). When such a trajectory is generated from a scalar signal \( x(t) \) through the time-delay embedding method, then the information which can be retrieved from URP\(_X\) depends on the choice of embedding dimension \( M \) and time-delay \( \tau \) as well as on the frequency content of the signal \( x(t) \) itself. Note that these results are also of importance for the information content of thresholded RPs, as these are constructed from URPs.

In Section 3 it is found that Fourier analysis offers an appropriate framework to analyze such questions. It is established that a URP always determines at least the power spectrum of the underlying signal \( x(t) \). For certain ‘unfortunate’ choices of \( M \) and \( \tau \) this may be all that can be retrieved from URP\(_X\). However, for better chosen values of \( M \) and \( \tau \) the URP may determine the signal \( x(t) \) up to a sign \( \pm 1 \). To analyze the effects of a specific choice of \( M \) and \( \tau \) we developed a graph theoretic approach, which also allows one to characterize the entire class of signals generating a given URP through the connected components of an associated graph \( G_X \). For signals with a known frequency content, this permits the computation of values for \( M \) and \( \tau \) which lead to maximally informative URPs. Such results are particularly applicable to discrete-time sampled signals, where the sample frequency limits the frequency content of the measurement signal. The examples of Section 6 demonstrate this. The first example illustrates the interplay between the frequency content of the signals and the values of \( M \) and \( \tau \) in determining the information content of the resulting URPs. The second example from EEG analysis concerns a sampled measurement signal to which similar considerations for \( M \) and \( \tau \) apply.

In Section 5 we addressed the actual reconstruction of a signal \( x(t) \) from a given unthresholded recurrence plot URP\(_X\). There we
focused on computing the Fourier coefficients \( c_k \) of \( x(t) \) directly from \( \text{URP}_X \) or the inner product function \((X(u), X(v))\). This can be achieved, up to the freedom admitted by the graph \( G_X \), as indicated in Proposition 5.2. An alternative approach is offered by the minimization of the distance measure \( F(X, Y) \) introduced in Section 4. This approach requires the selection of a parameterized class over which to minimize, and it needs a good starting point to avoid possible local minima. The latter is a disadvantage from a practical point of view. However, an important advantage lies in the possibilities to deal with noisy data and to reduce complexity by selecting an appropriate class for approximation. This is also demonstrated by the second example of Section 6. It shows that essential features of the measurement signal \( x(t) \) (which exhibits a Mu rhythm) are well preserved in the underlying signal \( y(t) \) when the URP is approximated according to this approach, provided that \( M \) and \( \tau \) are chosen adequately (to avoid approximation signals to occur such as \( x(t) \)).

A couple of research questions still remain open. (1) One important question concerns the accuracy of the reconstruction of \( x(t) \) from \( \text{URP}_X \). Clearly, for a continuous signal \( x(t) \) and a fixed embedding dimension \( M \), the unthresholded recurrence plot \( \text{URP}_X \) depends continuously on \( \tau \). Now let \( \tau = \tau_0 \) be a rational value for which \( G_X \) is disconnected and \( x(t) \) cannot be reconstructed up to a sign. As \( x(t) \) can in theory be reconstructed up to a sign for any irrational value of \( \tau \), it follows that such a reconstruction for values of \( \tau \) near \( \tau_0 \) cannot be numerically stable. Conversely, this implies that the URP obtained for values of \( \tau \) near \( \tau_0 \) cannot be very discriminative in representing certain information contained in \( x(t) \). From this perspective, the question of selecting appropriate values for \( M \) and \( \tau \) for which \( \text{URP}_X \) still contains all the information in \( x(t) \) (up to a sign), is only partially answered by the analysis of Section 3. In addition, if \( x(t) \) should in good approximation belong to a given parameterized class, then the minimization of \( F(X, Y) \), and an accompanying analysis of the stability or accuracy of that procedure, will shed light on the same issue, but from a practical point of view. (2) The selection of an adequate starting point for the minimization of \( F(X, Y) \) which avoids local (nonglobal) minima, is currently under investigation. (3) A third issue involves redundancy in the URP, which is a 2-dimensional representation of a 1-dimensional signal. The question arises to which extent a selected part of a URP may still contain all the information contained in the entire URP. This is of importance for relating subpatterns in an unthresholded recurrence plot \( \text{URP}_X \) to localized (morphological) properties of the underlying signal \( x(t) \).

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Appendix A. Proofs

Proof of Theorem 2.2. The ‘if’-part of this theorem easily follows by direct computation. To address the ‘only if’-part, let again \( \tilde{X}(t) = X(t) - X(t_0) \) and note that the affine basis \( \{X(t_0), X(t_1), \ldots, X(t_k)\} \) for \( V_X \) generates a (conventional) basis \( \{\tilde{X}(t_1), \ldots, \tilde{X}(t_k)\} \) for \( V_{\tilde{X}} \). Introduce \( B_X = (X(t_1) \cdots X(t_k)) \) which is an \( M \times k \) matrix of full column rank \( k \). Denoting the coordinate vector of \( \tilde{X}(t) \) by \( \alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_k(t) \end{pmatrix} \), it holds for all \( t \in \mathbb{T} \) that \( \tilde{X}(t) = B_X \alpha(t) \).

Note that \( \alpha(t_0) = 0 \), and for \( i \in \{1, \ldots, k\} \) we have \( \alpha_i(t_0) = e_i \), the \( i \)th standard basis vector in \( \mathbb{R}^k \). The inner product \( \langle \tilde{X}(u), \tilde{X}(v) \rangle \) attains the form \( \alpha(t)^T B_X^T B_X \alpha(u) \).

For the trajectory \( \tilde{Y}(t) \) and its associated subspace \( V_{\tilde{Y}} \), we carry out a similar construction. First, we let \( B_Y = (\tilde{Y}(t_1) \cdots \tilde{Y}(t_k)) \) and we consider points of the form \( B_Y \tilde{Y}(t) \). For \( t = t_0 \) we have that \( \alpha(t_0) = 0 \) and we arrive at the vector \( 0 = \tilde{Y}(t_0) \). For \( t = t_i \) with \( i \in \{1, \ldots, k\} \), we obtain \( B_Y \tilde{Y}(t_i) = \tilde{Y}(t_i) \).

In view of Lemma 2.1 with \( X_0 = X(t_0) \), if \( \langle \tilde{X}(u), \tilde{X}(v) \rangle = \langle \tilde{Y}(u), \tilde{Y}(v) \rangle \) for all \( u, v \) and, or equivalently \( \alpha(t)^T B_X^T B_X \alpha(u) = \alpha(t)^T B_Y^T B_Y \alpha(u) \), it then holds that \( B_X^T B_X \tilde{X}(u) = B_Y^T B_Y \tilde{X}(u) \), thus restricting \( \tilde{X}(u) \) and \( \tilde{Y}(u) \) to the set \( \{t_1, \ldots, t_k\} \). Since \( B_X \) has full column rank \( k \), the Gram matrix \( B_X^T B_X \) is \( k \times k \) invertible. Hence, also the \( M \times k \) matrix \( B_Y \) has full column rank \( k \). This implies that \( \dim(V_{\tilde{Y}}) \geq \dim(V_{\tilde{X}}) \).

However, by reversing \( X \) and \( Y \) and noting that they play symmetric roles, it follows likewise that \( \dim(V_{\tilde{X}}) \geq \dim(V_{\tilde{Y}}) \), so that
these dimensions do in fact coincide. Therefore \( \{ \overline{Y}(t_1), \ldots, \overline{Y}(t_k) \} \) constitutes a basis for \( V_{\overline{Y}} \).

We shall write \( \overline{Y}(t) = B_{\beta}(t) \) where \( \beta(t) \) denotes the coordinate vector of \( Y(t) \) with respect to this basis, for each \( t \in T \). Then \( \overline{Y}(u), \overline{Y}(v) \) attains the form \( \beta(v)^T B_{\beta}(u) B_{\beta}(v) \). Keeping \( u = t \in T \) fixed and varying \( v \) over the set \( \{ t_1, \ldots, t_k \} \), it follows that: \( c_i^T B_{\beta}(t) B_{\beta}(t) = c_i^T B_{\beta}(t) \) for all \( i \), implies \( B_{\beta}(t) \in \mathbb{C} \). This shows that \( \overline{Y}(t) = B_{\beta}(t) \) for all \( t \in T \).

It is a well-known result in linear algebra that \( B_{\beta}(t) B_{\beta}(t) \) is a unimodular constant. In this case therefore \( |X| = \overline{X} \).

Proof of Lemma 3.1. The inner product \( \langle T_p, T_q \rangle \) is given by a finite geometric series:

\[
\langle T_p, T_q \rangle = \sum_{m=1}^{M} 2^m - q = 2^p - q = 2^{p(q-q)} t_i.
\]

Hence it can be computed as:

\[
\sum_{m=1}^{M} 2^m - q = 2^{M/2} (2^M/2 - 2^{-M/2}) = 2^{M(q-q)} t_i,
\]

where \( \phi = \arg(z) = 2\pi(p-q) t_i \).

The rest of the lemma now follows.

Proof of Theorem 3.2. Note that the zero mean assumption on the periodic signals \( x(t) \) and \( y(t) \) implies that \( X = 0 \) and \( Y = 0 \) are corresponding points (i.e.: having identical coordinates) in the affine spaces \( V_X \) and \( V_Y \) spanned by the trajectories \( X(t) \) and \( Y(t) \), respectively. Thus, these spaces are in fact linear. In view of Lemma 2.1 it holds that \( URp = URp \) if and only if, and only if, \( \langle X, Y \rangle = \langle (U, 0), (0, Y) \rangle \). The latter identity between these two-variable functions implies that their 2D-Fourier representations, cf. Eq. (6), coincide.

Denoting \( x(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi i k t} \) and \( y(t) = \sum_{k \in \mathbb{Z}} d_k e^{2\pi i k t} \), it therefore holds for all \( p, q \in \mathbb{Z} \) that:

\[
c_p \bar{c}_q = \langle T_p, T_q \rangle = d_p \overline{d_q}(T_p, T_q).
\]

Choosing \( p = q \) and noting that \( \langle T_p, T_q \rangle = M \neq 0 \), we have \( |c_p|^2 = |d_p|^2 \) and the theorem follows.

Proof of Theorem 3.4. From the proof of Theorem 3.2 we have that \( URp = URp \) if and only if \( c_p \bar{c}_q = d_p \overline{d_q} \) for all \( p, q \in \mathbb{Z} \). As we have seen, this implies that the power spectra of \( x(t) \) and \( y(t) \) coincide, so that \( c_p = 0 \) if and only if \( d_p = 0 \). Therefore, the graphs \( G_X \) and \( G_Y \) coincide. Also, for each \( p \in K_X \) there exists a unique unimodular coefficient \( y_p \) such that \( d_p = y_p c_p \).

Suppose that \( \langle T_p, T_q \rangle \neq 0 \), so that \( p \) and \( q \) are adjacent in \( G_X \).

Then \( c_p \bar{c}_q = d_p \overline{d_q} \), which implies \( 1 = y_p \bar{y}_q \), whence \( y_p = y_q \) due to unimodularity. We conclude that every connected component of the graph \( G_X \) has its own unique associated unimodular constant.

As we are dealing with real signals only, it follows that two nodes \( p \) and \( q \) are adjacent if and only if the nodes \( -p \) and \( -q \) are also adjacent. Because \( c_{-p} = c_p \) and \( d_{-p} = d_p \), it follows that \( y_{-p} = \overline{y}_p \).

Now: if \( G_X \) is connected, then \( y_p = \overline{y}_p \) is identical for all \( p \in K_X \), so that \( y_p = \overline{y}_p = y \). Hence \( y_p = y \). Theorem 3.4. Conversely, if the graph \( G_X \) is not connected, then each connected component has its own unimodular constant. In this case freedom remains to vary the values of these constants (subject to the prescribed complex conjugacy requirements) without changing the URP.

Proof of Corollary 3.5. (1) In view of Lemma 3.1, it holds for all \( p, q \in K_X \) that \( (p-q)M_r \notin \mathbb{Z} \), so that \( \langle T_p, T_q \rangle \neq 0 \). Therefore \( G_X \) is a complete graph, hence connected.

(2) If \( M \) and \( d \) are co-prime, then \( (p-q)M_r \notin \mathbb{Z} \). Hence \( \langle T_p, T_q \rangle \neq 0 \). Therefore \( G_X \) is a complete graph, hence connected.

(3) If the denominator of \( M_r \) exceeds \( 2^k \), then \( (p-q)M_r \notin \mathbb{Z} \) for all \( p, q \in K_X \subseteq \mathbb{Z} \). Hence, all \( p, q \in K_X \) are adjacent and \( G_X \) is disconnected.

(4) Since \( M_r \in \mathbb{Z} \), any two nodes \( r \) and \( s \) are adjacent if and only if \( (r-s)M_r \in \mathbb{Z} \). In this case \( \langle T_p, T_q \rangle \neq 0 \). Therefore \( G_X \) is a complete graph, hence connected.

(5) If \( G_X \) is totally disconnected, then all connected components of \( G_X \) are automatically connected. Otherwise, suppose \( p \) and \( q \) are adjacent, hence in the same connected component of \( G_X \). Consider a node \( r \) in a different connected component; it exists because \( G_X \) is disconnected. Then \( \langle T_p, T_q \rangle \in \mathbb{Z} \). Hence, \( p \) and \( q \) adjacent implies \( \langle T_p, T_q \rangle \in \mathbb{Z} \). Conversely, \( \langle T_p, T_q \rangle \in \mathbb{Z} \) implies that \( p \) and \( q \) are adjacent.

If \( d \) denotes the denominator of \( r \), then \( \langle p, r \rangle \in \mathbb{Z} \) if and only if \( p \equiv q \mod d \). It follows that two nodes \( p \) and \( q \) are adjacent if and only if \( p \equiv q \mod d \). Therefore, for all \( p, q \in K_Y \), it holds that \( |p-q| \leq 2^k \). With \( M \in \mathbb{Z} \), adjacency of \( p \) and \( q \) is equivalent to \( \langle p, q \rangle \in \mathbb{Z} \), which requires \( p \) and \( q \) to belong to the same equivalence class modulo \( d \). If \( d > 2^k \) then distinct \( p \) and \( q \) can never be adjacent.

Proof of Corollary 3.6. (1) The choice of sign \pm 1 has no effect on the URP. Also, \( X(t) \) has the entries \( x(t+\pi), x(t+\pi), \ldots, x(t+(M-1)\pi) \) while \( x(t+\pi) \) has the entries \( x(t+\pi), x(t+(\pi+1)\pi), \ldots, x(t+(M+\ell-1)\pi) \). Because \( x(t) \) is periodic with period \( T = 1 \) and since \( M_r \in \mathbb{Z} \), it follows for example that \( \langle x(t+(M+\ell-1)\pi), x(t+(\pi+1)\pi), \ldots, x(t+(M+\ell-1)\pi) \rangle \) in fact identical to the entries of \( x(t) \), but cyclically shifted \( \ell \) positions. Therefore \( \langle X(u), x(v) \rangle = \langle x(u+\pi), x(v+\pi) \rangle \), for all \( u, v \in [0,1) \), establishing the result.

(2) One implication directly follows from Theorem 3.2. Conversely, if \( G_X \) is totally disconnected, and \( x(t) \) and \( y(t) \) have the same power spectrum, then \( |c_p| = |d_p| \) for all \( p \in \mathbb{Z} \). For all such \( x(t) \) that \( c_p \neq 0 \) (where \( d_p \neq 0 \)), the unimodular coefficients \( y_p \) follow uniquely from \( y_p = d_p/c_p \). Since \( c_p = d_p \) and \( d_p = \overline{d_p} \), also \( y_p = \overline{y_p} \). Therefore the coefficients \( y_p \) satisfy all the indicated requirements, so that \( URp = URp \).

(3) Starting from \( URp = URp \), it follows that for all \( k \in K_X \) there exist unique unimodular coefficients \( y_p \) such that \( d_p = y_p c_p \). Identical \( URp \) then equivalently require that \( c_p \bar{c}_q = d_p \overline{d_q} \) for all \( p, q \in \mathbb{Z} \). If \( c_p \bar{c}_q \neq 0 \), it follows equivalently that \( y_p = y_q \). Now, the multiplicative relationship \( d_k = y_k c_k \) in \( K \subseteq \mathbb{Z} \) in the Fourier domain, is well known.
to be equivalent to a convolution relationship in the time domain: $y(t) = (x \ast y)(t)$ for $y(t) = \sum_{k=-\infty}^{\infty} y_k e^{2\pi i k t}$. The fact that $x(t)$ and $y(t)$ are real implies that $y(t)$ is also real.

Conversely, let the Fourier coefficients of $y(t)$ be required to have the properties $|y_k| = 1$ for all $k \in \mathbb{K}$, and $y_p = y_q$ if $p$ and $q$ are adjacent in $G_X$. Then $y_p = y_q$ if $c_p c_q(T_p, T_q) \neq 0$, whereas $y(t) = (x \ast y)(t)$ yields $d_k = y_k c_k$. As a result $URP_X = URP_Y$. □

**Proof of Proposition 4.3.** The distance measure $F(X, Y)$ is given explicitly by

$$F(X, Y) = \int_{0}^{1} \int_{0}^{1} \left( \|X(u) - X(v)\|^2 - \|Y(u) - Y(v)\|^2 \right)^2 \, du \, dv.$$ 

First, the integrand is expanded as:

$$\left( \|X(u)\|^2 - 2\langle X(u), X(v) \rangle + \|X(v)\|^2 - \|Y(u)\|^2 \right)^2 + 2\left( \langle Y(u), Y(v) \rangle - \|Y(v)\|^2 \right)^2.$$ 

Regrouping the terms in this expression it is then expanded into a sum of six terms as:

$$\left( \|X(u)\|^2 - 2\langle X(u), X(v) \rangle + \|X(v)\|^2 - \|Y(u)\|^2 \right)^2 - 2\left( \langle X(u), X(v) \rangle - \langle Y(u), Y(v) \rangle \right)^2$$

$$= \left( \|X(u)\|^2 - \|Y(u)\|^2 \right)^2 + \left( \|X(v)\|^2 - \|Y(v)\|^2 \right)^2$$

$$+ 4\left( \langle X(u), X(v) \rangle - \langle Y(u), Y(v) \rangle \right)^2$$

$$+ 2\left( \|X(u)\|^2 - \|Y(u)\|^2 \right)^2 \left( \|X(v)\|^2 - \|Y(v)\|^2 \right)^2$$

$$- 4\left( \|X(u)\|^2 - \|Y(u)\|^2 \right) \left( \|X(v)\|^2 - \|Y(v)\|^2 \right) \langle X(u), X(v) \rangle - \langle Y(u), Y(v) \rangle \rangle.$$

The double integrals of the first two terms together yield the expression $2N_1(X, Y)$. The double integral of the third term yields $4I(X, Y)$. The double integral of the fourth term can be separated into a product of two single integrals, yielding the expression $2N_2(X, Y)$. Finally, the double integrals of the fifth and sixth terms both vanish. This follows from linearity of the inner product (in both arguments) and the zero-mean assumption on the trajectory, allowing, e.g., to rewrite any double integral of the form $\int_{0}^{1} \int_{0}^{1} f(u) X(u) X(v) \, du \, dv$ as $(\int_{0}^{1} f(u) X(u) \, du) (\int_{0}^{1} X(v) \, dv)$, where $\int_{0}^{1} X(v) \, dv = 0$. □

**Proof of Proposition 5.1.** If one integrates $URP_X(u, v)^2$ with respect to the variable $v$ only, it is obtained that

$$\int_{0}^{1} URP_X(u, v)^2 \, dv = \int_{0}^{1} \left( \|X(u)\|^2 - 2\langle X(u), X(v) \rangle + \|X(v)\|^2 \right)^2 \, dv$$

$$= \|X(u)\|^2 + \|X(v)\|^2.$$

Here it is used that $\int_{0}^{1} \int_{0}^{1} f(u) X(u) X(v) \, du \, dv = \langle X(u), \int_{0}^{1} f(u) X(u) \, du \rangle = 0$ due to the zero mean assumption on the signal. If one integrates $URP_X(u, v)^2$ with respect to both variables $u$ and $v$, then:

$$\int_{0}^{1} \int_{0}^{1} URP_X(u, v)^2 \, du \, dv = \int_{0}^{1} \left( \|X(u)\|^2 \right) \, du + \int_{0}^{1} \left( \|X(v)\|^2 \right) \, dv$$

$$= 2 \int_{0}^{1} \|X(t)\|^2 \, dt.$$

The inner product $\langle X(u), X(v) \rangle$ can be expressed in terms of trajectory norms and $URP_X(u, v)^2$ as:

$$\{ X(u), X(v) \} = \frac{1}{2} \left( \|X(u)\|^2 + \|X(v)\|^2 - URP_X(u, v)^2 \right).$$

Using the previous two identities to express $\|X(u)\|^2$ and $\|X(v)\|^2$ in terms of integrals of $URP$s, the result follows. □

**Proof of Proposition 5.2.** The coefficients of the 2D-Fourier representation

$$\{ X(u), X(v) \} = \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} c_p c_q e^{2\pi i (pu - qv)} \langle T_p, T_q \rangle$$

can be computed from the inner product $\langle X(u), X(v) \rangle$ in the usual way, by exploiting unitarity of the complex 2-dimensional harmonic basis $\{ e^{2\pi i (pu - qv)} \mid p, q \in \mathbb{Z} \}$.

$$c_p c_q(T_p, T_q) = \int_{0}^{1} \int_{0}^{1} \{ X(u), X(v) \} e^{-2\pi i (pu - qv)} \, du \, dv.$$ 

This proves Eq. (17), provided that $c_p c_q(T_p, T_q) \neq 0$, i.e., $p$ and $q$ are adjacent. By choosing $p = q = k$, so that $\langle T_p, T_q \rangle = M \neq 0$ and $e^{-2\pi i (pu - qv)} = e^{-2\pi k (u - v)} - i \sin(2\pi k (u - v))$, while $\langle X(u), X(v) \rangle$ and $|c_k|^2$ are real, one may address just the real part of this identity to arrive at Eq. (16):

$$|c_k|^2 = \frac{1}{M} \int_{0}^{1} \int_{0}^{1} X(u), \langle X(v) \rangle \cos(2\pi k (u - v)) \, du \, dv.$$ 

Repeated application of Eq. (17) to the pairs of adjacent nodes in the path $k = n_0 \to n_1 \to \cdots \to n_{\ell} \to -k$ gives:

$$\prod_{i=1}^{\ell} Q(n_{i-1}, n_i) = \frac{c_{n_0}}{c_{n_1}} \frac{c_{n_1}}{c_{n_2}} \cdots \frac{c_{n_{\ell-1}}}{c_{n_\ell}} = \frac{c_n}{c_{-k}} = c_{-k}.$$ 

Multiplication by $P(k) = c_k c_{-k}$, the fact that $c_{-k} = c_k$ since $x(t)$ is real, Eq. (18) follows. □

**Proof of Proposition 5.3.** This is a direct consequence of unitarity of the complex 2-dimensional harmonic basis $\{ e^{2\pi i (pu - qv)} \mid p, q \in \mathbb{Z} \}$, analogous to Parseval’s theorem in one dimension, together with the definition of $F(X, Y)$ as the squared 2-norm of the function $URP_X(u, v)^2 - URP_Y(u, v)^2$, which has the 2-dimensional Fourier coefficients $\{ c_{pq} - D_{pq} \mid p, q \in \mathbb{Z} \}$. □

**References**