Cross recurrence quantification of coupled oscillators

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Abstract

Subtle nonlinear behaviors of fluid-coupled mechanical oscillators at low and medium viscosities were better detected by cross recurrence analysis than spectral analysis. Cross recurrence with its high sensitivity to nonlinear dynamics may have applicability to weakly coupled oscillators prevalent in biology and physiology.

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1. Introduction

In 1665 Christian Huygens discovered that pendulum clocks mounted on a common wall would eventually swing in synchrony (see [1]). This observation marked the beginning of the study of mutual dependency among relatively independent components that is generally referred to as coupling. Coupling between signals is typically quantified by conventional methods such as relative phase relations, coherence analysis, or cross-correlation. However, these tools are linear and assume that individual components have additive mutual influences, rather than nonlinear multiplicative interactions. Many natural systems, physical and biological, are nonlinear, nonstationary and noisy, and thereby violate many of the assumptions of traditional linear methods. Cross-recurrence analysis (CRQ) is posited as a more robust, less assumptive measure of coupling than conventional methods.

Recurrence quantification analysis (RQA) was originally designed by Webber and Zbilut [2, 3] to study the recurrent structuring of single signals that were time-delayed and embedded in higher-dimensional space. These auto recurrence plots were demonstrated to have utility in diagnosing the states of a variety of dynamical systems [4]. Cross recurrence analysis (CRQ) was introduced by the same authors [5] to exam-
ine the intricate recurrent structuring between paired signals which were also time-delayed and embedded in higher-dimensional space. Similarly, Marwan et al. [6] have most recently studied the synchronization of paired time series (mathematical sine waves) and paired spatial series (geophysical sediment cores) using cross recurrence plots. In the present study the dynamics of simpler physical systems, coupled and uncoupled mechanical oscillators, were examined using CRQ with performances compared against standard spectral analysis (FFT). Of interest was the greater or lesser ability of these tools to detect subtle changes in coupling strength effected by altering the viscosity of the coupling medium. As is discussed, beyond mere physical systems, CRQ may have important potential applicability to numerous types of coupled oscillators in biological and physiological systems.

2. Methods

2.1. Apparatus

A dual-oscillator system was constructed according to a modified design of von Holst [7] who modeled physiologically coupled oscillators with a mechanical analog. Our mechanical model consisted of two oscillators which were coupled via a fluid medium as diagrammed in Fig. 1 (top). The first oscillator, the driver, consisted of an electric rotary motor which imparted reciprocating motion to a plastic tray (∼20 cm wide, ∼30 cm long, ∼12 cm deep). The tray was filled with fluids of varying viscosity into which the second oscillator was submersed. The second oscillator, the rotor, was an aluminum rod with a wooden spindle on the top (air phase) and a spherical paddle on the bottom (fluid phase). The spindle was wound with a nylon string which was attached, via a series of pulleys, to a vertically suspended weight (150 g). Release of the weight and the pull of gravity caused the spindle to unwind and the rotor to spin within the viscous fluid medium of the driven tray. Displacement (cm) of each oscillator was recorded using a Fastrack magnetic tracking system with motion capture software (Skill Technologies). Magnetic sensors were placed on recording levers attached to each oscillator to reproduce their displacement positions (Fig. 1, bottom).

2.2. Procedure

The two oscillators (driver-tray and follower-rotor) were coupled via three fluids with differing viscosities (quantified below): syrup/honey (high viscosity); syrup (medium viscosity); syrup/oil (low viscosity). The volume of fluid was identical in each situation. When both the tray and rotor were set into motion concurrently, the system was defined as coupled though the viscous medium. When only the rotor or tray was in motion, the system was defined as uncoupled even though the paddle was submersed in the fluid medium. At each viscosity level, four replicate trials were run for each of three couplings: driver motion with rotor motion (coupled); rotor motion without driver motion (uncoupled, type 1); driver motion without rotor motion (uncoupled, type 2). After the rotor came up to speed (elimination of transients), 10 seconds of oscillator displacement(s) were digitized at 60 Hz (600 points per run). The average period and frequency data of the coupled and uncoupled oscillators are provided in Table 1. These data were all computed from the duration of peak-to-peak displacement cycles occurring within each 10 second window (see Fig. 1). To quantify relative viscosity values, the mean periods of the free-running rotor oscillator (uncoupled, type 1) were normalized to the periods of the highest viscosity. As shown, these normalized (unitless) values worked out to be as follows: 1.000 (high viscosity), 0.711 (medium viscosity), and 0.693 (low viscosity). It is important to note that, by design, the two lower viscosities were only within 1.8% of each other, whereas they differed from the highest viscosity by 28.9% or 30.7%. This minimal difference between the low viscosities became a test on the ability to detect subtle, nonlinear changes in oscillator couplings in this range.

2.3. Data analysis

Each 10-second segment of recorded amplitude (displacement) data from the driver and rotor oscillators were individually normalized over the unit interval (see Fig. 1). These normalized files were subsequently analyzed by three methods: spectral analysis; return plots; cross-recurrence analysis. First, spectral analysis was performed on the 600-point files using a 512-point fast Fourier transform (FFT). Spectral power values were expressed in arbitrary, but identical
Fig. 1. Schematic diagram for two mechanical oscillators, powered by a sinusoidal driver or falling weight, and coupled through a viscous fluid medium (top). Normalized positions of tray (independent driver) and rotor (dependent follower) with coupling fluids of high, medium or low viscosity (bottom). The data are digitized at 60 Hz for 10 seconds.

units which allowed for fair comparisons of spectra between the different viscosity runs. Data were conditioned by Hanning windowing (10 percent cosine tapering), the dc component was removed, but linear detrending was neither necessary nor performed. The frequency range from 0 to 3 Hz was examined with a fixed resolution of 0.117 Hz per bin (60 Hz/512 points).

Second, return plots of the rotor data were generated to examine the influence of viscosity on the shape of the dynamic attractors in coupled and uncoupled situations. These qualitative graphs were easily obtained
Table 1
Frequency and period data of coupled and uncoupled driver and rotor oscillators. Data are expressed as mean ± stdev for four 60-second trials

<table>
<thead>
<tr>
<th>Coupling type</th>
<th>Driver frequency (Hz)</th>
<th>Driver period (s)</th>
<th>Rotor frequency (Hz)</th>
<th>Rotor period (s)</th>
<th>Relative viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>coupled</td>
<td>0.625 ± 0.001</td>
<td>1.599 ± 0.002</td>
<td>0.626 ± 0.003</td>
<td>1.597 ± 0.007</td>
<td>syrup/honey</td>
</tr>
<tr>
<td>uncoupled-type 1</td>
<td>no motion</td>
<td>no motion</td>
<td>1.292 ± 0.031</td>
<td>0.775 ± 0.019</td>
<td>1.000</td>
</tr>
<tr>
<td>uncoupled-type 2</td>
<td>0.626 ± 0.001</td>
<td>1.598 ± 0.002</td>
<td>no motion</td>
<td>no motion</td>
<td>high viscosity</td>
</tr>
<tr>
<td>coupled</td>
<td>0.621 ± 0.004</td>
<td>1.611 ± 0.009</td>
<td>1.347 ± 0.106</td>
<td>0.746 ± 0.059</td>
<td>syrup</td>
</tr>
<tr>
<td>uncoupled-type 1</td>
<td>no motion</td>
<td>no motion</td>
<td>1.817 ± 0.018</td>
<td>0.551 ± 0.005</td>
<td>0.711</td>
</tr>
<tr>
<td>uncoupled-type 2</td>
<td>0.624 ± 0.007</td>
<td>1.604 ± 0.018</td>
<td>no motion</td>
<td>no motion</td>
<td>medium viscosity</td>
</tr>
<tr>
<td>coupled</td>
<td>0.621 ± 0.006</td>
<td>1.611 ± 0.016</td>
<td>1.581 ± 0.029</td>
<td>0.633 ± 0.012</td>
<td>syrup/oil</td>
</tr>
<tr>
<td>uncoupled-type 1</td>
<td>no motion</td>
<td>no motion</td>
<td>1.861 ± 0.006</td>
<td>0.537 ± 0.002</td>
<td>0.693</td>
</tr>
<tr>
<td>uncoupled-type 2</td>
<td>0.622 ± 0.004</td>
<td>1.608 ± 0.009</td>
<td>no motion</td>
<td>no motion</td>
<td>low viscosity</td>
</tr>
</tbody>
</table>

by plotting relative rotor positions as delayed versions of themselves. To open up the attractors, a delay of 0.167 s (lag of 10 points sampled at 60 Hz) was found to be satisfactory in each case. These attractors appeared as Lissajous figures.

Third, cross recurrence analysis (CRQ) was implemented to correlate driver and rotor positions in coupled and uncoupled situations (program KRQI). An $N \times N$ distance matrix ($D$) was computed from the driver (dri) and rotor (rot) time series using an embedding dimension ($m$) of 5 and delay ($\tau$) of 0.017 s (1/60 Hz or lag of 1 time point), such that

$$D_{i,j} = \sqrt{\sum (d_{i+m\tau} - rot_{j+m\tau})^2}. \quad (1)$$

The embedding dimension was set to 5 in order to capture the two-dimensional system (driver and rotor oscillators) with noise (Hegger et al. [8]). The time delay vectors were constructed with lags of 1 to assure the highest temporal resolution between the two time signals operating at different frequencies. This is in accordance with the fact that there is no optimal lag (delay) for non-stationary systems (Grassberger et al. [9]). Distances were rescaled to the maximum distance in the matrix and therefore ranged from 0% to 100%. Recurrent points were located at coordinates $i, j$ whenever distances between specific $dri_{i+m\tau}$ and $rot_{j+m\tau}$ vectors fell within a predefined radius (Heaviside exclusion threshold). With low values of radius, the recurrence matrix was necessarily sparse. Since $m = 5, N = 596$ for each 600-point time series. The optimal radius was determined systematically (see Section 3). Because the $dri_{i+m\tau}$ and $rot_{j+m\tau}$ time series were dissimilar, cross recurrence plots were necessarily asymmetric across an imaginary central diagonal (absent). Thus all CRQ plots contrasted with the ubiquitous symmetry across a prominent central diagonal for auto recurrence plots characterized by self-identity matching. As originally defined [2], three quantitative features were extracted as recurrence variables from the cross recurrence plots. First, percent recurrence ($\%REC$) was defined as the density of recurrence points within the $N \times N$ recurrence matrix. Second, percent determinism ($\%DET$) was defined as the fraction of recurrent points forming diagonal line structures of length two or greater. Third, the longest of these line structures, MAXLINE, was a measure of the longest parallel trajectory maintained between the coupled oscillators. Program KRQI was used to compute the elapsed time between cross recurrent points. The shortest time interval (0.017 s) was discarded as noise, but all subsequent intervals (0.033–9.917 s) were converted to frequency values (0.101–30 Hz) and binned in a histogram. Bin counts were plotted as functions of frequency up to 3 Hz for alignment with the spectral plots. But unlike the fixed bin resolution of spectral analysis, the KRQI bin resolution necessarily increased nonlinearly as the recurrent frequency decreased (see Section 3 for details). Similar methods have been previously implemented by the authors in detecting pre-ictal patterns in EEG signals using auto-RQI [10].

3. Results

The driver oscillator had a fixed frequency of 0.623 Hz (0.621–0.626 Hz range) independent of ei-
Fig. 2. Frequency analysis of oscillator dynamics at three different coupling viscosities: high, medium, or low. With increases in viscosity, the frequency of the uncoupled rotor decreased slightly (dotted lines), whereas the frequency of the coupled rotor decreased greatly (solid lines). At the highest viscosity, the diver oscillator captured the coupled rotor oscillator at the frequency of the sinusoid (vertical dashed lines).

Fig. 2 shows the frequency analysis of the oscillator dynamics under different coupling viscosities. The high viscosity case shows a decrease in frequency for the uncoupled rotor (dotted line) and a significant decrease for the coupled rotor (solid line). At the medium viscosity, the uncoupled rotor's frequency decreased further, and the coupled rotor's frequency continued to decrease. In the low viscosity case, the uncoupled rotor's frequency decreased to a much lower value, and the coupled rotor's frequency further decreased. The diver oscillator captured the coupled rotor oscillator at the frequency of the sinusoid (vertical dashed lines).

The driver was robust, stable, and unchanging in its oscillatory characteristics, making it an ideal forcing function for this study. All attention, therefore, was focused on the dependent rotor oscillator whose frequency characteristics from the linear FFT and nonlinear KRQI perspectives are shown in Fig. 2. With increasing viscosity, the natural frequency of the uncoupled rotor (dotted lines) decreased from 1.861 Hz to 1.817 Hz to 1.292 Hz (Table 1). However, coupling of the two oscillators through the viscous medium (solid lines),
caused a slowing of the rotor rotations from 1.581 Hz to 1.347 Hz to 0.626 Hz dependent upon the viscosity (Table 1). At the highest viscosity, the rotor was fully captured and synchronized by the driver (same oscillatory frequencies).

It is very important to compare the FFT and KRQI spectra (Fig. 2). First, over the frequency range from 0 Hz to 2.5 Hz, the frequency resolution is far better for KRQI (572 histogram bins) than for FFT (22 histogram bins). Only at frequencies from 2.5 to 3.0 Hz is the frequency resolution better for FFT than KRQI (4 histogram bins each), but none of the experimental data fell within this range. Second, KRQI and FFT give essentially the same information in uncoupled situations irrespective of the fluid viscosity. This is to be expected since the system is fully linear when the rotor is oscillating at its natural frequency with no interference from the driver. Third, coupling of the rotor to the driver through the low and medium viscosity fluids causes a splaying out of KRQI frequencies not clearly observed with the FFT. There are even subtle differences between the KRQI spectra for these two viscosities that are nonetheless very close to one another (0.693 and 0.711; Table 1). These KRQI data reveal nonlinear coupling characteristics that are completely missed by linear FFT analysis. Fourth, only when the driver and rotor oscillators are coupled one to one in the highest viscous fluid (1.000; Table 1) do the FFT and KRQI show agreement once again, save for the much narrower peak due to the finer resolution of KRQI. These results are due to the restored linear coupling between the paired oscillators in which all nonlinear interactions seen at lower viscosities have been extinguished.

Similar points can be made from the return plots pictured in Fig. 3. For each of the three uncoupled situations, the rotor defined three limit-cycle attractors (Fig. 3(D),(E),(F)). These tight ovals are observed at each viscosity and are consistent with the linear nature of this single oscillator, undisturbed by the competing driver oscillator which was quiescent. Coupling the two oscillators together, however, greatly perturbs the geometry of the attractors. For the two lower viscosities, the attractors display quasi-periodicities which are essentially indistinguishable from one another (Fig. 3(B),(C)). The complicated patterns result from nonlinearities in the viscous couplings. At the highest viscosity the attractor returns to a limit cycle (Fig. 3(A)). Its geometry is not a simple oval; rather the trajectories form a more complicated shape, but they still retrace (overlay) one another.

The driver and rotor data were subjected to cross recurrence analysis. Setting the embedding dimension to 5 and the lag to 1 point (delay of 0.017 s) meant that trajectory profiles of the driver and rotor dynamics could be compared consecutive five points at a time. When these short trajectories were close in $N$-space, by definition they were said to recur. The question is how close? This is a threshold question relating to the proper setting of the radius parameter. Fig. 4 illustrates how the radius was selected at 2% of the maximum distance of the recurrence matrix. In each case, coupled and uncoupled oscillators, the radius was systematically increased in steps of 0.1% while computing $\%$REC values. The $\%$REC was then plotted against radius on double-log scales. As can be seen, with increases in radius, more and more recurrent points are recruited. But at radius values of 2%, the $\%$REC values were all close to 1%, assuring that the recurrence matrix was sparse and that only vectors close in $N$-space were registered as recurrent, yet not so sparse as to lose statistical relevancy. Nevertheless, at each viscosity level, the $\%$REC was consistently higher in the coupled versus uncoupled situation.

Cross recurrence plots are displayed in Fig. 5 for coupled and uncoupled situations for all three viscosities. Although not shown, the driver tray time series proceeds from left to right beneath each horizontal $i$-axis, and the rotor time series proceeds from bottom to top at the left of each vertical $j$-axis. Because these two time series are never identical, no central diagonal ever appears in any plot. For the uncoupled cross recurrence plots (Fig. 5, right panels), the rotor dynamics (uncoupled type 1) were paired with the driver dynamics (uncoupled type 2) recorded in separate runs. All causal correlations were precluded and the recurrent points form regular, uninteresting dot-like patterns with low $\%$REC values. As expected, the vertical distancing between these recurrent points defines recurrent intervals (which are expressed as reciprocals in Fig. 2). For the coupled cross recurrence plots (Fig. 5, left panels), the rotor dynamics were paired with the driver dynamics recorded in the same runs. These correlations are therefore causal, resulting in slightly higher $\%$REC, especially at the highest viscosity. These coupled cross recurrent points form far
Fig. 3. Return plots of the relative rotor position as a function of itself delayed. Strong, limit-cycle attractors are formed for each of the uncoupled situations (D), (E), (F) as well as the high viscosity, coupled situation (A). Low (C) and medium (B) viscosity coupling induces quasiperiodic, nonlinear behavior in the rotor dynamics. These coupled data correspond exactly to the time series plotted in Fig. 1.

more interesting patterns than their uncoupled pairs. Thus for the low viscosity, the points form global patterns with an upward slant of 20 degrees and occasional solid line tails at 45 degrees. For the medium viscosity, the global slant increases to 25 degrees and line structures at 45 degrees form short heads and long tails at 45 degrees. For the high viscosity, the global pattern is brought into 45-degree alignment with central swellings and long head and tail lines also at 45 degrees slant. These graphs indicate the qualitative sensitivity of cross recurrence plots to detect subtle nonlinearities in coupled oscillators as influenced by small changes in viscosity (e.g., low to medium viscosity) as well as large changes in viscosity (e.g., medium to high viscosity).

But qualitative plots are not robust against personal biases, necessitating the quantification of the cross recurrence plots. Three features were extracted from each plot and quantified as %REC, %DET, and MAX-LINE (defined in Section 2). These CRQ variables are graphed as functions of relative viscosity in Fig. 6 for both coupled (solid lines) and uncoupled (dashed lined) situations. For coupled oscillators, a very small change in fluid viscosity (low to medium) resulted in
Fig. 4. Determination of the optimal radius parameter for cross recurrence analysis. %REC values are recomputed as the radius parameter was incremented in steps of 0.1% from 0.0% to 5.0% of the maximum distance in the recurrence matrix (double log plots). The radius was selected at 2% (dashed line) which corresponded to low %REC values (< 1%) for each situation, save the highest viscosity in the coupled case (> 1%).

Four replicate trials are shown for each condition.

4. Discussion

Cross recurrence quantification is posited as a nonlinear tool to gauge the degree of coupling between simultaneously recorded signals. The advantage of cross recurrence analysis as compared to more conventional methods (e.g., spectral analysis, cross correlation, coherence analysis, etc.) is that CRQ requires no assumptions regarding the size, distribution, linearity or stationarity of the data under scrutiny. In this Letter a model of minimal complexity was exploited to illustrate how CRQ reveals subtle interactions between oscillators with various coupling strengths. Even though the driver oscillator was a sine wave generator by design (linear), the motions of the coupled rotor oscillator were much more complex and delicate (nonlinear). Thus CRQ was able to tease out non-obvious dynamic characteristics of the weak couplings (low and medium viscosities) not assailable by spectral analysis or return maps. Only in the case of strong, one-to-one coupling (high viscosity) did FFT and CRQ reach similar conclusions. This was due to the collapse of the system’s nonlinear complexity into simple linear interactions. Stated in another way, when the fluid viscosity was high, there was no “slippage in the clutch”. Here the linear motion of the driver was effectively passed on to the follower rotor, rendering its dynamics linear also. Thus the advantage of CRQ over FFT surfaces only when the oscillator couplings are weak and nonlinear in nature. This is analogous to the enhanced sensitivity of auto-recurrence over FFT in detecting higher-dimensional nonlinearities in non-stationary fatiguing muscles [11].

There are many details surrounding the implementation of CRQ, the most important of which is the scal-
Fig. 5. Qualitative cross recurrence plots of coupled oscillators (left panel) and uncoupled oscillators (right panel) at high viscosity (top row), medium viscosity (middle row), low viscosity (lower row).
this sense, CRQ can be used as a transition detector on paired signals in much the same way auto-recurrence analysis can be used to detect transitions along single trajectories [2,11].

Secondly, it should be recognized that CRQ variables (%REC, %DET, MAXLINE) are all computed within the recurrence window irrespective of the phasic relationships between paired signals. For the mechanically coupled oscillators in this Letter the recurrence window was appropriately large (596 points or 9.93 s) because the oscillator dynamics were studied in their steady states. Steady states were favored by discarding the initial transients as the rotor came up to speed, and proven by the uniformity of recurrent points in all cross recurrence plots. However, to assess the phase relationships between two signals that are in transition states, it is possible to compute CRQ on an episodic or moving window basis (e.g., program KRQE). The resolution of the phasing is set by the duration of the window; the resolution of the recurrence variables is set by the offset between windows which can be as fine as one point (1/digitization frequency). Future studies can explore the utility of windowing the paired oscillators to examine phase relationships in transition states.

Thirdly, CRQ methodology can be applied to any dual oscillators regardless of the mode of their coupling. For example, the coupling could be linear or nonlinear, strong or weak, with or without integer frequency ratios, stationary or nonstationary. This flexibility and utility stems from the recurrence methodology itself which requires no a priori assumptions re-
Regarding the input time series. Thus, paired recurrences between two time series are merely tallied without any requirements for transforming (distorting, filtering) the data.

The principle motive of this study, akin to that of von Holst [7], was to give entry into the field of biological oscillators. Future studies will be required to study these more complex systems, but the CRQ results herein are non-trivial and robust in their demonstrated effectiveness in detecting subtle dynamics of weakly coupled, albeit simple oscillators. By useful extension, CRQ is hypothesized as being critically important in investigating biological systems that are notoriously rhythmic, nonlinear, non-stationary, and noisy. Except for special clinical conditions (e.g., grand mal seizures), the couplings between biological oscillators are typically weak, unstable, non-robust and state-dependent (e.g., cardio-respiratory oscillators). The couplings between biological systems can also be decidedly transient, making it impossible to use standard techniques on the signals which depend upon statistical properties of the inputs (i.e., the data sets are too abbreviated). Thus it is suggested that CRQ may overcome these limitations and be useful in detecting subtle and transitory couplings between biological oscillators be they of neural, mechanical, hormonal, or humeral nature, to name a few. Specifically, coupling strength between biological oscillators has been shown to be critically important in plasmiodal slime molds [12], leech swim oscillators [13], locomotor central pattern generators [14], phase transitions in human hand movements [15], critical fluctuations of rhythmic movements between people [16], and emergent properties of coupled nonlinear oscillators [17]. Since each of these systems was analyzed by standard linear tools, reanalysis with nonlinear tools like CRQ should give deeper insight into the complex dynamics unfolding in weak coupling modes.

5. Conclusions

Cross recurrence analysis is a nonlinear tool that can detect subtle nonlinearities between paired mechanical oscillators that are weakly coupled through a viscous medium. The coupling characteristics can be described either qualitatively using cross recurrence plots or quantitatively using cross recurrence variables. CRQ is promoted as an analysis tool that should be useful for the study of weakly coupled biological oscillators from a new perspective.

6. Software

Cross recurrence programs KRQD, KRQL, KRQS, and KRQE used for data analysis or alluded to in this Letter are available from http://homepages.luc.edu/~cwebber/ in self-extracting file format (RQA62.EXE).

References