Transient chaos in a globally coupled system of nearly conservative Hamiltonian Duffing oscillators

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Abstract

In this work, transient chaos in a ring and globally coupled system of nearly conservative Hamiltonian Duffing oscillators is reported. The networks are formed by coupling of three, four and six Duffing oscillators. The nearly conservative Hamiltonian nature of the coupled system is proved by stability analysis. The transient phenomenon is confirmed through various numerical investigations such as recurrence analysis, 0–1 test and Finite Time Lyapunov Exponents. Further, the effect of damping and the average transient lifetime of three, four and six coupled schemes for randomly generated initial conditions have been analyzed. The experimental confirmation of transient chaos in an illustrative system of three ringly coupled Duffing oscillators is also presented.

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1. Introduction

A large number of experimental as well as theoretical investigations have been made on coupled systems and networks. This is because, coupled systems and networks, have a wide range of applications ranging from physics and engineering to economy and biology. Transient chaos is a phenomenon exhibited by deterministic nonlinear dynamical systems, wherein trajectories starting from randomly chosen initial conditions look chaotic up to long time length, and then quite abruptly switch over into a final periodic state [1]. The transient chaos is essentially a regime, whereby the distance between a strange attractor and boundary of its basin of attraction in the phase space decreases until they touch each other at a critical value of the control parameter \( p = p_c \). At this point, the chaotic attractor exhibits a crisis (and ceases to exist at \( p > p_c \)), converting into an unstable chaotic manifold called chaotic saddle. Situation is transient chaos is due to a non-attracting chaotic saddle in the phase space.

The Hamiltonian system is disturbed through the small amount of perturbation, the system may be possible to give a transient chaos [2]. Kantz and Grassberger investigated the chaotic transient in many deterministic systems [3]. This phenomenon is often observed in many theoretical, numerical simulation and experimental studies such as in hydrodynamic experiments [4], radio circuits [5], chemical reactions [6], optical bi-stable media [7], distributed systems of the type of an electron flow interacting with a backward wave [8], and standard models of the nonlinear dynamics such as logistic map [9], Eno map [10], and Rössler system [11], parametrically damped pendulum [12], NMR laser experiments [13], the experimental model of the nonlinear pendulum (mechanical model) [14]. It has also been reported in neural networks, wherein a dynamical structure similar to the Hopf field neural network exhibits transient behavior which converges to stable equilibrium points. This dynamical system is called as transient chaotic dynamics in neural network (TCNN) [15]. A four dimensional autonomous chaotic oscillator which exhibits the transient chaos under particular conditions has been reported by Chang et al. [2010] [16]. The transient chaotic behavior is closely connected with many other aspects of nonlinear dynamics like scattering process,
diffusion and other transport phenomena [17,18]. Recently, a noise induced system which exhibits a interesting phenomenon of transient chaos has also been reported [19].

While transient chaos in individual systems is by itself an interesting phenomenon, its presence in coupled systems or a small world network of oscillators has opened up exciting possibilities. This is because small world network of oscillators have been used to mathematically model the social factors such as the internet, power grid, pattern of neuron connectivity, and even a network of movie actors and actresses [20–22]. So, the collective behavior of the dynamical systems, and globally coupled oscillators have received attention during recent years [23–25]. While much numerical studies and analytical studies have been done on transient chaos, very few experimental works on transient chaos exhibited by individual as well as coupled system have been reported. In fact, the motivation of the present work is to address this lacuna. Investigating transient chaos in dynamical system is quite important because in numerical investigation one observes such behavior very often. Most of the cases the dynamics presented in numerical investigation is not confirmed in experiments due to the parameter mismatch introduced by real elements. In this paper, we give the clear experimental results which matches closely with the numerical results. This experiment extend the feasibility of observing transient chaos in real physical systems. Also, we have used recent techniques in nonlinear dynamics used to characterize transient chaos. To our knowledge, we believe that it is for the first time the phenomenon of transient chaos is investigated both through numerical studies and hardware experiments for three coupled oscillator. This work is in continuation of our earlier paper, where we reported that a system of two bi-directionally coupled system have been reported. In this paper, we give the clear experimental results as given below,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -b x_2 - \omega x_1 - \beta x_1^3 + \varepsilon (x_3 + x_5 - 2x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -b x_4 - \omega x_3 - \beta x_3^3 + \varepsilon (x_1 + x_5 - 2x_3) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -b x_6 - \omega x_5 - \beta x_5^3 + \varepsilon (x_1 + x_3 - 2x_5).
\end{align*}
\]  

(2)

where each oscillator is indexed by \(i = 1, 2, \ldots, N\). Here, \(b\) is the damping co-efficient, \(\omega\) is the natural oscillating frequency, \(\beta\) is the strength of nonlinearity, and \(\varepsilon\) is the coupling co-efficient, and the values are \(b = 0.0001\), \(\omega = -0.35\), \(\beta = 0.85\) and \(\varepsilon = 0.06\) are fixed. \(A_{ij}\) is an adjacency matrix (\(a_{ij} = 1\), if units \(i\) and \(j\) are connected, and 0 otherwise) [28,29]. In all the nodes of the oscillators kept in double well potential \((\omega < 0, \beta > 0)\). Fig. 1, shows the schematic diagram of three, four and six globally coupled oscillators drawn using Pajek software. The number indicates the single oscillator and the double headed arrow refers to the systems coupled diffusively.

3. Stability analysis

In order to investigate the stability of the system, we consider the globally coupled model of Duffing oscillators. For the three coupled oscillators shown in Fig. 1(a), we consider a set of six first order coupled normalized equations as given below,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -b x_2 - \omega x_1 - \beta x_1^3 + \varepsilon (x_3 + x_5 - 2x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -b x_4 - \omega x_3 - \beta x_3^3 + \varepsilon (x_1 + x_5 - 2x_3) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -b x_6 - \omega x_5 - \beta x_5^3 + \varepsilon (x_1 + x_3 - 2x_5).
\end{align*}
\]

(2)

To find the equilibrium points of Eq. (2), we set the coupling term \(\varepsilon\) and the damping co-efficient \(b\) to zero, and fix the parameter values as \(\omega = -0.35, \beta = 0.85\) Further setting \((x_{1,2,6}) = 0\), the equilibrium points are obtained. They are \(x_{1,15} = 0, \pm \sqrt{\left|\omega\right|/\beta}\), \(x_{2,4,6} = 0\). The Jacobean matrix \(J\) of Eq. (2) is written as,

\[
J = \begin{bmatrix}
-\lambda & 1 & 0 & 0 & 0 & 0 \\
-\alpha - 3\beta x_1^2 - 2\varepsilon & \lambda & 0 & \varepsilon & 0 & 0 \\
0 & 0 & -\lambda & 1 & 0 & 0 \\
\varepsilon & 0 & -\alpha - 3\beta x_3^2 - 2\varepsilon & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & -\lambda & 1 \\
\varepsilon & 0 & \varepsilon & 0 & -\alpha - 3\beta x_5^2 - 2\varepsilon & \lambda & \\
\end{bmatrix}
\]

(3)

Evaluating the Jacobian matrix, the eigenvalues \((j_{1,2,6})\) are obtained. Here, we consider three cases of damping co-efficient \(b = 0.0, \pm 0.0001\). For \(b = 0\), the eigenvalues are \(\pm i 0.5916\) the perturbation neither decays to zero nor diverges to infinity but it varies periodically with time. The trajectories do not approach the equilibrium point as \(t \to \infty\). The equilibrium point is called center type or elliptic equilibrium point and is neutrally stable. The eigenvalues are, \(-0.0001 \pm i 0.5916\) for \((b > 0)\), the trajectory wind around the equilibrium point a number of times before
reaching it asymptotically. In this case, the equilibrium point is called a stable spiral point or stable focus. Similarly for \((b < 0)\), the trajectories move away from the equilibrium point along spiral paths. The equilibrium point is an unstable focus in this case. The eigenvalues \((\lambda_1)\) in this case are \(0.0001 \pm i0.5916\). The eigenvalues of the three coupled system in Fig. 1(a), plotted as the function of damping coefficient \(b\) is shown in Fig. 2. We find that in both the cases of damping, the real part of the eigenvalues are too small, namely \(\pm 0.0001\). This low eigenvalues indicates that the asymptotic behavior (whether stable or unstable) will be reached after a long time. Hence the three coupled system which is a nearly conservative Hamiltonian system, becomes dissipative and exhibits transient chaos in the process. In a similar manner, we calculate the eigenvalues for the case of four and six globally coupled oscillatory network. The results are tabulated in Table 1.

4. Numerical results

In this section, we describe the results of our numerical simulations. Let us consider the smallest number \(N = 3\) of Eq. (1) Duffing oscillators as given in Fig. 1(a). For positive damping, the system converges asymptotically to a fixed point, while for negative damping, the system becomes exponentially divergent [26]. To numerically simulate Eq. (2), we use Runge-Kutta 4th order method with a fixed step size 0.01. The dynamics of the system can be studied qualitatively by plotting the trajectories in the two dimensional \((x(t) - \dot{x}(t))\) phase plane. Fig. 3a(i), shows that the phase plane of the \((x_1(t) - x_2(t))\) variable of Eq. (2), red color indicates the transient behavior and the blue color indicates that the asymptotic behavior of the system i.e periodic. Fig. 3a(ii) shows that the time evolution of the \((x_1(t))\) variable of Eq. (2). In the Poincaré map construction, the motion from one crossing with a secant surface until the next one corresponds to the phase shift \(2\pi\). When we consider coupled autonomous chaotic systems, we still can construct a partial Poincaré map, e.g., taking successive maxima of the variables partial frequencies are then simply defined as an average number of crossings of the secant surfaces per unit time [30]. According to this approach, the maxima of the points were collected in the \((x_1(t), x_2(t))\) variable of Eq. (2), for every oscillations. The geometric structure of the transient chaos in the Poincaré surface of section appears as a totally disconnected and uncountable set of points (red dots). The densely interwoven points which gives the periodic oscillations (blue dots) is shown in the Fig. 3b(i). The time series can also be seen by visual inspection that the transient behavior related to chaotic saddles up to 2400 data points and then the dynamics becomes regular (inset) is shown in the Fig. 3b(ii).

4.1. Effect of the damping co-efficient \((b)\)

The damping forces acting on the individual systems, seem to affect the transient chaotic behaviors. This is shown by calculating the average transient lifetime \((\tau)\) of the transient chaotic attractor as a function of the damping co-efficient \((b)\) (The details of the calculating the average transient lifetime is shown in Section 5). This average transient lifetime for particular damping co-efficient \(b\) was obtained by simulating the system for a 100 set of initial conditions selected randomly. These simulations were repeated for the entire range of the damping co-efficient \(b\), while keeping the other parameters as constants, with \(\omega = -0.35, \beta = 0.85\). The results are plotted in the Fig. 4. The average transient lifetimes for three, four and six glob-
Table 1

<table>
<thead>
<tr>
<th>Equilibrium points</th>
<th>( b = 0.0 )</th>
<th>( b = 0.0001 )</th>
<th>( b = -0.0001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Three coupled scheme:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x_1, x_3) = 0.0)</td>
<td>(\pm 0.5916i)</td>
<td>(\pm 0.0001 \pm 0.5916i)</td>
<td>(\pm 0.0001 \pm 0.5916i)</td>
</tr>
<tr>
<td>((x_3, x_5) = \pm \sqrt{</td>
<td>\omega</td>
<td>}/\beta)</td>
<td>(\pm 1.1832i)</td>
</tr>
<tr>
<td><strong>Four coupled scheme:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x_1, x_3, x_5) = 0.0)</td>
<td>(\pm 0.8367i)</td>
<td>(-0.00005 \pm 0.8367i)</td>
<td>(-0.00005 \pm 0.8367i)</td>
</tr>
<tr>
<td>((x_3, x_5) = \pm \sqrt{</td>
<td>\omega</td>
<td>}/\beta)</td>
<td>(\pm 0.9695i)</td>
</tr>
<tr>
<td><strong>Six coupled scheme:</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((x_1, x_3, x_5, x_7) = 0.0)</td>
<td>(\pm 0.8650i)</td>
<td>(-0.00005 \pm 0.8650i)</td>
<td>(-0.00005 \pm 0.8650i)</td>
</tr>
<tr>
<td>((x_3, x_5, x_7) = \pm \sqrt{</td>
<td>\omega</td>
<td>}/\beta)</td>
<td>(\pm 0.9169i)</td>
</tr>
</tbody>
</table>

Fig. 3. a(i) Phase portraits and b(i) Poincaré surface of section in the \((x_1(t) - x_2(t))\) plane of stabilization of periodic (blue color) from transient chaos (red color) with the initial conditions \(x_1 = 0.03, x_3 = 0.003, x_{2,4,6} = 0.1\) of the fixed parameters \(\omega = -0.35, \beta = 0.85\) and \(\varepsilon = 0.06\) of Eq. (2). The corresponding time waveform are shown in Fig. a(ii) and b(ii). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. Damping co-efficient \((b)\) vs average transient lifetime \((t)\) for averaging 100 set of randomly chosen initial conditions and the other system parameters are fixed as \(\omega = -0.35, \beta = 0.85\) and \(\varepsilon = 0.06\). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

ally coupled Duffing oscillator systems of Fig. (1) are shown in red, pink and blue color respectively. From Fig. (4), we find that when the system is completely conservative, all the three global networks have almost infinite lifetimes indicating that they are exhibiting permanent chaos. However for small values of damping co-efficient, the lifetimes begin to decrease. This is because the global networks are nearly conservative and after sufficiently long time settle to periodic behavior. It is in this parametric regime, that transient chaos is observed. When damping becomes high, all the global networks settle down straight away to periodic behavior. For our numerical simulations we have taken the damping value as \(b = 0.0001\), because it is for this value, the transient lifetime is sufficiently long for our analysis.

4.2. Phase \((\phi)\) and Poincaré return map

To elucidate the transient dynamics, the instantaneous phase \((\phi)\) of the time series is calculated [31] by using the Hilbert transformation. Given a signal \(x_1(t)\), its amplitude \((A)\) and phase \((\phi)\) of the analytical or complex signal \(\psi(t)\) can be determined through the equation

\[
\psi(t) = x_1(t) + i\tilde{x}_1(t) = A(t)e^{i\phi(t)}
\]

where,

\[
\tilde{x}_1(t) = \frac{1}{\pi} P.V. \left( \int_{-\infty}^{+\infty} \frac{x_1(t')}{t-t'} dt' \right).
\]
Here the $P.V.$ denoting the Cauchy principal value in the Hilbert transform. The phase can be unwarped by tracing the $\approx 2\pi$ jumps in the time course of $\phi(t)$. In the course of calculating the HT, we have adopted the techniques given in the Ref. [36]. Fig. 5(a) shows the phase of the signal $x_1(t)$ which is generated using Eq. (2). It can be seen that the phase $\langle \phi \rangle$ increases monotonically (inset) as the time increases and settles to a constant value. The increase in phase confirms the presence of transient chaos. The saturation of phase to a constant value, confirms the regular or periodic behavior of the system. This makes impossible the observation of the recurrence of the dynamical states in higher dimensional (two or three) phase spaces can be pictured. Phase spaces of higher dimension (say $m$), can only be visualized in an abstract sense. This makes impossible the observation of the recurrence of the dynamical states in higher dimensional phase spaces. However, the Eckmann’s [36,33] idea of projecting higher dimensional phase spaces into the two or three dimensional subspaces enables one to investigate the $m$ dimensional phase space trajectory through a two dimensional representation of its recurrences. Such recurrence of a state at time $i$ at a different time $j$ is marked within a two dimensional square matrix whose columns and rows correspond to a pair of time scales and the elements are marked as ones and zeros (picted as black and white dots in the plot). From the multivariate $x = x_1, x_2, \ldots x_m$ time series data, the data corresponding to $x_1(t)$ variable alone is taken and the dynamics is reconstructed onto a lower $n$ dimensional phase space using a time delay $(\tau)$. The reconstructed trajectory $Z$ can be expressed as a matrix where each row is a phase space vector given by,

$$Z = [y_1, y_2, y_3 \ldots y_n]^T,$$

where, $y_i = [x_i, x_{i+1}, \ldots x_{i+(D_k-1)T}]$ and $m = n - (D_k - 1)T$, where $D_k$ is the embedding dimension and $T$ is the delay time. Any recurrence of state $i$ with state $j$ is pictured on a Boolean matrix expressed by [34],

$$R^D_{ij} = \Theta(\psi - |y_i - y_j|),$$

where, $\Theta(.)$ is the Heaviside function, $\psi$ is the arbitrary threshold, and $y_i \psi R^D_{ij}$ is the embedding system. To confirm the presence of transient chaos as well as the asymptotic periodic behavior, we make use of the Recurrence Plots. For this, we use 10000 data points of a single variable $x_1(t)$ obtained numerically using RK4 algorithm with a fixed step size of 0.01. The time series for this data set was plotted and the duration of chaotic and periodic states of the system were obtained. Then the data sets corresponding to chaotic and periodic behavior were separated. Using these two data subsets for the variable $x_1(t)$, the dynamics in the reduced phase space were reconstructed assuming the embedding dimension as $D_k = 6$ and time delay $\tau = 76$. From this reconstructed phase space, the Recurrence Plots corresponding to the $x_1(t)$ variable for chaotic and periodic regimes are obtained. They are shown in Fig. 6. In Fig. 6a(i) and b(i), the time series plots for the chaotic and periodic behavior are shown respectively. Corresponding to these behaviors the Recurrence Plots are shown in Fig. 6a(ii) and b(ii). Figs. 6a(iii) and b(iii) are blown up portions of Figs. 6a(ii) and b(ii) for clarity. The random distributions of the dots in Fig. 6a(ii) while the diagonal distributions of the dots in Fig. 6b(ii) clearly differentiate the transient chaos from the periodic behavior.

4.3. Recurrence analysis

A Recurrence Plot (RP) is a technique employed in nonlinear data analysis. It is a graphical visualization of the repetitions or recurrences of a particular state of the system as its dynamics evolves. Generally only low dimensional (two or three) phase spaces can be pictured. Phase spaces of higher dimension (say $m$), can only be visualized in an abstract sense. This makes impossible the observation of the recurrence of the dynamical states in higher dimensional phase spaces. However, the Eckmann’s [36,33] idea of projecting higher dimensional phase spaces into the two or three dimensional subspaces enables one to investigate the $m$ dimensional phase space trajectory through a two dimensional representation of its recurrences. Such recurrence of a state at time $i$ at a different time $j$ is marked within a two dimensional square matrix whose columns and rows correspond to a pair of time scales and the elements are marked as ones and zeros (picted as black and white dots in the plot). From the multivariate $x = x_1, x_2, \ldots x_m$ time series data, the data corresponding to $x_1(t)$ variable alone is taken and the dynamics is reconstructed onto a lower $n$ dimensional phase space using a time delay $(\tau)$. The reconstructed trajectory $Z$ can be expressed as a matrix where each row is a phase space vector given by,

$$Z = [y_1, y_2, y_3 \ldots y_n]^T,$$

where, $y_i = [x_i, x_{i+1}, \ldots x_{i+(D_k-1)T}]$ and $m = n - (D_k - 1)T$, where $D_k$ is the embedding dimension and $T$ is the delay time. Any recurrence of state $i$ with state $j$ is pictured on a Boolean matrix expressed by [34],

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4.4. 0–1 test

The 0–1 test is a binary test to classify periodic or chaotic states proposed by Gottward and Melbourne [35,36]. In this approach, the dynamics of the system is plotted in a space of translation variables and the average growth

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**Fig. 5.** (a) Phase ($\phi$) of the variable $x_1(t)$ computed through numerically of Eq. (2), with the initial conditions $x_{1,5} = 0.03, x_3 = 0.003, x_{2,4,6} = 0.1$ of the fixed parameters $\omega = -3.5, \beta = 0.85$ and $\varepsilon = 0.06$ using Hilbert transformation, (b) Poincaré return map plotted in $x_1(t)$ variable in the $(x_n - x_n+1)$ plane of 10000 data points with fixed step size of 0.01.
rate of the mean square displacements of trajectories in this translation variable space, expressed as $K$ is taken as a characteristic measure of the dynamics. It is found that $K$ takes on a value 1 for chaos and a 0 value for periodic behavior. The translation variables are taken as

$$p(n) = \sum_{j=1}^{n} x(j)\cosjc$$

$$q(n) = \sum_{j=1}^{n} x(j)\sinjc,$$

where $c$ is a randomly chosen constant varying in the range $(\pi - 2\pi)$, and $x(j)$ is the time series for any chosen variable of the system. In the space of the translation variables, a bounded motion refers to a periodic state while a random Brownian like motion indicates a chaotic state.

Using these translation variables, the mean square displacement $M(n)$ is calculated as

$$M(n) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} [p_{c,j+n} - p_{c,j}]^2 + [q_{c,j+n} - q_{c,j}]^2.$$
This mean square displacement $M(n)$ grows exponentially for chaotic behavior while it varies periodically for regular periodic state. The asymptotic growth rate $K_c$ of the mean square displacement $M(n)$ is calculated from the equation,

$$K = \lim_{n \to \infty} \frac{\log M(n)}{\log n}.$$  \hspace{1cm} (10)

While for transient chaos $K \approx 1$ it will take on a zero value for periodic behavior. To perform this ‘0–1’ binary test, two sets $n = 400$ each of data points were taken from the chaotic and periodic regions of time series. The graphical representation was made in the space of translation variables as shown in Fig. 7(a). From this representation, the transient chaotic behavior $a(i)$ and the periodic behavior $a(ii)$ got differentiated clearly. While the trajectories show a Brownian like Motion for transient chaos, they show bounded motion for periodic behavior. The mean square displacement of the trajectories for these two cases are shown in $b(i)$ and $b(ii)$ respectively. The $K$ value plotted as a function of discrete time for both the cases are shown in $c(i)$ and $c(ii)$. We get $K = 0.9908$ ($\approx 1$) for transient chaos and $K = 0.00032$ ($\approx 0$) for periodic state.

4.5. Finite time Lyapunov exponent

The finite time Lyapunov exponent (FTLE) is a statistical measure of the amount of stretching (or contraction) of a trajectory along a particular direction over a finite time interval. The FTLE is expressed as [37].

![Fig. 7. 0–1 test of numerically computed transient time trajectory of the $(x_1(t))$ variable in Eq. (2), of chaotic segment and periodic segments which separated from the transient time series, (a) Dynamics of the translation components ($p(n), q(n)$), (b) mean square displacement ($M(n)$), and (c) asymptotic growth rate ($K$) for the (i) chaotic and (ii) periodic state.]
\[ z^m_k = \frac{1}{M} \sum_{j=1}^{M} z^m_j, \quad m = 1, 2, \ldots k. \]  

(11)

where, \( M \) denotes the time interval, and \( z^m_k \) is the instantaneous Lyapunov exponent which is defined as,

\[ z^m_k = \log |e^m||, \quad m = 1, 2, \ldots k. \]  

(12)

The reorthonormalization vector \( e^m \) denoted as,

\[ e^m_j = JM(x_j, y_j, z_j, \Theta_j, \phi_j) \varepsilon^m_j. \]  

(13)

\( JM \) is the Jacobian matrix, and \( j \) refers to the time step. Here we have calculated the FTLE for Eq. (2), fixing the parameters as \( \omega = -0.35, \beta = 0.85 \) and \( \varepsilon = 0.06 \) and the initial conditions as \( x_0 = 0.03, x_3 = 0.003. x_{2,4,6} = 0.1. \) The variations of the FTLE's, calculated using \( M = 100 \) data points, as functions of time are shown in Fig. 8. For the value of \( t \) \((t < 0.24 \times 10^4)\), three exponents are positive and remaining are negative. When the time \( t \) is increased above \((t > 0.24 \times 10^4)\) all the three largest Lyapunov exponent become to zero while the sum of all Lyapunov exponents becomes negative. This shows that the system exhibits transient chaos/hyperchaos and transits to periodic behavior.

In the transient chaos/hyperchaos regime, we find that the distribution of transient chaos is a function of initial conditions for a distinct value of \( \varepsilon = 0.06 \). This is illustrated in Fig. 9. Here we have plotted the phase diagram in the \((x_2 - t)\) plane. Fixing the initial conditions of the variables \( x_1 \) and \( x_{3-6} \) as earlier, the initial values for \( x_2 \) were changed in the range \((-2.0 \text{ to } 2.0)\). For these initial conditions the Lyapunov exponents were calculated using a total of \( 50,000 \) data points. Using these Lyapunov exponent values, the periodic behaviors were separated from the chaotic saddles. For this, we assumed the value \( (\lambda_1 = 0 \text{ or } 0.00001) \) to denote periodic behavior and the values \( (\lambda_1 \geq 0.00001) \) as representing chaotic saddle. In figure, the regions containing chaotic saddles are denoted as (CS) while the region showing periodic behaviors are denoted by (P). These two regions are differentiated online by different colors: blue for chaotic saddles and red for periodic orbits. The probability of the transient lifetime is obtained taking the averaging of the entire range of the transient time.

5. Average transient lifetime

A quantitative measure of how long the transient chaos exists is given by the “average transient lifetime” [38]. There are several methods to calculate the average transient lifetime of the trajectories before they exit the transient regime [39,40]. In this paper, we determine the lifetimes of the trajectories while the system exists in transient chaos/hyperchaos regime by calculating the Finite Time Lyapunov Exponents. The lifetime of transient chaos/hyperchaos is taken as the duration of time form zero to the instant when the largest Lyapunov exponent value becomes less than or equal to \( (\lambda_1 = 0 \text{ or } 0.00001) \). The procedure is repeated for 100 set of randomly generated initial conditions and the average of these values is taken.

Fig. 10, gives the average transient lifetime \( (t) \) vs coupling co-efficient \( (\varepsilon) \) of (blue) three, (Pink) four and (Red) six coupled schemes. From the figure, we can see that as \( \varepsilon \) decreases, the transient lifetime becomes longer up to \( (\varepsilon = 0.025) \) which is equal to the system is very near to the unsynchronized. We compute Eq. (1), for \( N = 3,4,6 \) for three, four and six oscillators with fixed parameter as \( b = 0.0001, \omega = -0.35, \beta = 0.85, \) and \( \varepsilon = 0.06. \) Fig. 11 shows the log-log plot (the monomial equation: \( y(x) = kx^p \), and the logarithmic of the equation base \( e \), as \( \log(y) = k\log(x) + \log a \) of power law distribution using the data where plotted in Fig. (10). In the Fig. (11) obeys the power law behavior with \( k = -0.304, b = 8.221 \) for three, \( k = -0.866, b = 6.471 \) for four, and \( k = -1.023, b = 5.202 \) for six oscillator respectively for the average transient lifetime data.

6. Experimental results

In this section, we give the real time hardware experimental circuit realization of three ring coupled Duffing
oscillators and their coupled dynamics. The circuit is as shown in Fig. 12. Here each individual oscillator is by the red frame of dotted lines. Each oscillator is built using two capacitors \((C_1, C_2)\), six linear resistors \((R_{12...6})\) and two multipliers \((AD633/\text{N})\). The multipliers are used to generate the nonlinearity term \(x^3\). The output of the multipliers are, 
\[ W = (x_1 - y_1)(x_2 - y_2)/10 + Z, \]
where \(x_1, x_2, y_1, y_2\) are the input signals, \(W\) is the output and \(Z\) is a correction of the output signal [41]. The circuit equations obtained using Kirchhoff’s laws. For example the first oscillator block \((v_1, v_1)\) as,
\[
\frac{d^2 v_1}{dt^2} = \frac{1}{R_1 C_1} \frac{dv_1}{dt} - \frac{v_1}{R_4 R_5 C_1 C_2} + \frac{0.01 v_1^3}{R_2 R_5 C_1 C_2} + \frac{1}{R_4 R_5 C_1 C_2} \left( \frac{R_{13}}{R_{12}} v_3 + \frac{R_{13}}{R_{11}} v_2 - \frac{R_{13}}{R_{10}} v_1 \right).
\]
Similarly, the circuit equation obtained by the nodes \((v_2, v_2, v_3, v_3)\). Assuming the normalized values of voltages \(v(t)\) as \(x(t)\) and re scaling the various parameters as \(t - \tau = R_4 C_1 t, b = R_d/R_1, \omega_0^2 = R_4/R_5, \beta = 0.01 R_4/R_6\), and the coupling co-efficient \(e = R_4/R_5\), the normalized equations for the Duffing oscillator for the double well case given Eq. (2) can be obtained. The components values are chosen as, \(R_1 = 10\Omega, R_{12...4}, R_{12...6}, R_{11...13} = 10K\Omega, R_5 = 28.5K\Omega, R_6 = 117\Omega, R_7 = 116.6K\Omega, R_{10} = 5K\Omega\) and the capacitance values are \(C_1, C_2 = 10 nF\) the accuracy is \(\pm 1\) percentage of tolerance.

The time series of the voltage \(v_1(t)\) across the capacitor \(C_1\), is obtained using AGILENT INFINIVISION MSO6014A series oscilloscope. Experimentally observed transients time series are shown in the Fig. 13(a), the transient regime exist up to 2800 mS and then the circuit reaches its steady state as periodic. Further, to confirm the robustness of the observed transient time and the influence of noise in the circuit, we have calculated the Signal to Noise Ratio (SNR). For the intrinsic noise the SNR have been

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**Fig. 10.** Average transient lifetime \(t\) of the chaotic transient for three (blue line), four (pink line) and six (red line) globally coupled Duffing oscillators vs coupling co-efficient \(\epsilon\). In each data was obtained by averaging of 100 set of randomly chosen initial conditions of Eq. (1) of \(N = 3.4.6\) with the fixed parameter values \(b = 0.0001, \alpha = -0.35, \beta = 0.85, \) and \(\epsilon = 0.06\). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Fig. 11.** Log–log plot of power law distribution using the data where plotted in Fig. 10. The power law behavior with \(k = -0.304, b = 8.221\) for (a) three, \(k = -0.866, b = 6.471\) for (b) four, and \(k = -1.023, b = 5.202\) for (c) six coupled oscillators.
calculated as 22.4233 dB. Also we have checked the robustness of the experiment with external noise. While applying the external noise of 10 mV and 30 mV the system exhibits transient chaos. Therefore as our experiment is robust against external noise. Also, the result shows a high qualitative agreement between the numerical (see Fig. (3)) and the experimental results. The Fig. 11(b) shows phase ($\phi$) calculated for the experimentally observed time series using Hilbert transformation. Both the time series plot and the calculated value of phase are in good agreement with numerical results.

7. Summary

In this work, we have presented the interesting phenomena of transient chaos in a system of three, four and six globally coupled nearly conservative Hamiltonian Duffing oscillators. We have confirmed the presence of transient chaos numerically using various statistical measures. In addition, we present the experimental evidence of transient chaos. The average transient lifetime calculated for all the three cases in globally connected schemes are found to obey a power law. Also when the
coupling co-efficient $\varepsilon$ decreases, the transient lifetime becomes longer depending on the number of the systems that are coupled.

Acknowledgments

The authors would like to acknowledge Prof. Tamás Tél for his valuable comments and suggestions, which greatly improved the quality of this paper, and also we thank A. Ishaq Ahamed for a critical reading of the manuscript. S.S. acknowledges the University Grants Commission (UGC) for the financial assistance through RFSMS scheme. K.T. acknowledges DST, Govt. of India for the financial support through the Grant No. SR/S2/HEP-15/2010.

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