Optical complexity in external cavity semiconductor laser

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\textbf{A B S T R A C T}

In this article, the window based complexity and output modulation of a time delayed chaotic semiconductor laser (SL) model has been investigated. The window based optical complexity (OC), is measured by introducing the recurrence sample entropy (SampEn). The analysis has been done without and in the presence of external noise. The significant changes in the dynamics can be observed under induced noise with weak strength. It has also been found that there is a strong positive correlation between the output power and the complexity of the system with various sets of parameters. The laser intensity, as well as the OC can be increased with the incremental noise strength and the associated system parameters. Thus, optical complexity quantifies the system dynamics and its instabilities, since is strongly correlated with the laser outputs. This analysis can be applied to measure the laser instabilities and modulation of output power.

1. Introduction

During recent years, semiconductor lasers (SL)\textsuperscript{[1,2]} have been a potential area of theoretical and experimental investigations\textsuperscript{[3,4]}. A SL model with external cavities conveys various dynamical phenomena due to the changes of its key parameters like feedback strength, feedback delay, pumping current, etc. The dynamics can be described by the nature of instabilities in a delayed SL system. The theoretical model corresponding to the dynamics of a single mode SL has been introduced in 1980\textsuperscript{[5]}. One of the interesting applications of such lasers is chaos-based optical communications\textsuperscript{[6,7]} in which chaotic signals can be effectively used to encrypt a message and improve privacy and security in data transmission. These systems define the dynamics, demonstrate a chaotic regime when subjected to chaotic oscillations by injection from another source\textsuperscript{[8]}, reflection from an external mirror\textsuperscript{[9]}, and also changes in the other associated parameters. It also has been reported that the output power (laser intensity) can be modulated\textsuperscript{[10]} by changing the associated parameters and under the influence of an additive noise with significant noise efficiency. In\textsuperscript{[11,12]}, the results reveal that the size of the external cavity plays a crucial role in optimizing the output power when subjected to optical feedback. The effect of additive Gaussian noise on the dynamics of an SL has been a subject of active research for the past few decades. Gaussian noise plays a stabilizing and destabilizing role on the dynamics and also exhibits a wide variety of noise induced phenomena. Of late, it has been reported that the use of Gaussian noise is not always appropriate as it is unbounded and there exists a non-zero probability for Gaussian noise for having very large values. This causes an unexpected complicated change in the dynamics of an SL. Non-Gaussian bounded noise can play a vital role in the study of the dynamics of a model system with known parameter values causing random fluctuations. However, this kind of noise has rarely been reported due to the lack of mathematical tools so as to obtain analytical results. In our previous analysis\textsuperscript{[13]}, we also reported the effect of Sine-Wiener noise\textsuperscript{[14,15]}, a non-Gaussian bounded noise having zero mean and stationary correlation function.

The induced noise effect can be investigated well with $\frac{1}{\sqrt{\tau}}$ noise. It can show various color spectrum as well as flicker noise with varying $\beta$. A brownian motion can also have a spectrum with $\beta=2$. The effect of $\frac{1}{\sqrt{\tau}}$ can also be observed in several real experiments and natural phenomenon like earthquakes, experimental results on sea level and bermuda etc.
Complexity is a measure which quantifies the amount of instability in a system. In fact, it can measure the complex agreement of state variables in a dynamical system [16–19]. It has been studied that such unpredictable behavior can be quantified by information entropy, introduced by Shanon [20]. However, various entropy measures like Permutation entropy [21–25], Sample entropy (SampEn) [26,27], Kolmogorov-Sinai (K-S) entropy [28,29], Renyi entropy [30,31], spectral entropy (SE) [32] can be used to measure the complexity of a system. In [33] it has been observed that modified SampEn can successfully quantify the complexity of a single mode SL system. Since entropy is a statistical invariant measure, it is obvious that such invariant information does not possess similar trend in each partition of the phase space of a system. We have introduced a window based measure - optical complexity (OC) by implementing the concept of SampEn. Windowing is a kind of scaling technique which samples a signal with different time resolutions. Various methods can be used to scale a signal [34–36]. Multi-scaling is one of the promising tools in nonlinear time series analysis [34–37]. In our article, a uni-scale windowing technique [36] has been applied in windowing the phase space of the SL system. The implementation is done by modification of aforesaid SampEn with recurrence based window structure of the phase space.

It has been noted in theoretical works that the OP can be maximized when the laser is in a chaotic state [11]. In the external cavity SL, the light produced by a laser diode is directed back into its active layer upon reflections by the external mirrors. The optical delay generated by the external cavity time may result in strongly nonlinear behavior and chaos [38,39], that may eventually constructively affect the OP. In case of laser diodes, external currents also produce similar instabilities [40]. For instance, in the case of Lorenz-Haken phenomenon, a high gain and low line-width is required to achieve the instabilities in the dynamics. Haken suggested pumping the laser 20 times more than the first conventional threshold [41,42], to get a second one, above which one can achieve strong nonlinearity and the Lorenz Haken chaos. The power enhancement related to the chaotic regimes, that has been theoretically predicted and numerically simulated, has been experimentally observed as well [43]. In particular, lasers diodes have been found indeed to be highly sensitive to optical feedback and to enjoy much higher output in the chaotic states than in periodic states. At the same time, the OC defined in terms of recurrence based measures, behaves similarly to the largest Lyapunov exponent [29,44], which implies that whenever the system is in periodic or multi-periodic state, it is relatively low, compared to its values in the chaotic states. In turn, the window based Sample entropy extracts information about the dynamics and its instabilities in phase space. Since the dynamics (intensity) and the instability (complexity) are associated by the information generation, they have a strong correlation.

The manuscript is organized as follows: In Section 2 we investigated the dynamics of a delayed SL-model with two external cavities and in the presence of external noise. It has been observed that the periodic structures transferred to be chaotic, in the presence of the external noise. The dynamics has been quantified by single and two parameter bifurcation diagrams transferred by 0–1 test [45]. The 0–1 test is based on the measure ‘mean square displacement’ (MSD), from the diffusive and non-diffusive part of a time series. For regular dynamics, the MSD is a bounded function of time; whereas as it scales linearly with time in case of chaotic state. The asymptotic growth ($K_c$) of MSD can be the measure to quantify the dynamics of a system or a time series. It can be applicable for a deterministic as well as stochastic dynamics [46]. The value $K_c$ close to 1 and 0 indicate chaotic and regular dynamics respectively. Significant changes can be observed in the dynamics of the SL model under the effect of external noise. Section 3 introduces the measure OC, which is based on windowing sample entropy. The changes of the complexity with various parameters in the presence of noise are also investigated in details. The analysis of output power, its modulation and its relation to the optical complexity has also been observed. Section 5 is the conclusion.

2. Dynamics of two delayed semiconductor laser

2.1. Deterministic and noise induced SL model

The dynamics of a SL has been investigated by several theoretical models. Fig. 1, represents the schematic diagram of a SL with two external cavities. In the schematic diagram, the beam splitter [47,48] can be the dielectric mirror to split the incident light beam. Any partially reflecting mirror can be used for splitting light beams. The angular separation of the output beams, we can have any positive value. Different power splitting ratios can be achieved via different designs of the dielectric coating. Apart from the dielectric mirrors, there are other kind of splitters such as splitter cubes, fiber-optic beam splitters, metal coated mirrors [47,48]; to split the beams with various separation angles. The SL concerned with optical feedback can be described by a set of coupled delay differential equations-a dimensionless LK model [49,50]. We generalized the set by introducing two different delays $\tau_1$ and $\tau_2$, which can be written as

$$\frac{dE}{dt} = 0.5(1 + i\omega)(n-1) + n_1 E_0(t-\tau_1)e^{-i\omega t} + n_2 E_0(t-\tau_2)e^{-i\omega t},$$

$$\frac{dn}{dt} = \frac{1}{T}(n-0.5(n-1) + nE^2),$$

(1)

where $E_0(t)$ and $n(t)$ are intracavity complex electric field and carrier population, respectively; $\alpha$ is the line width enhancement factor; $\omega$ is the frequency of the solitary laser and ($n_1$, $n_2$) are the feedback rates. For photon lifetime $\tau_p$ and carrier lifetime $\tau_n$, $T$ is defined as $T = \tau_p/\tau_n$ is proportional to the pumping rate above threshold. Since the electric field amplitude $E$ and phase $\phi$ holds the relation: $E_0 = Ee^{i\phi}$, the eq. (1) can be written as

$$\frac{dE}{dt} = 0.5E(n-1) + n_1 E(t-\tau_1)\cos(\phi - \phi(t-\tau_2) + \omega_2),$$

$$\cos(\phi - \phi(t-\tau_2) + \omega_2),$$

$$\frac{dn}{dt} = \frac{1}{T}(n-0.5(n-1) + nE^2),$$

(2)

By perturbing the system (2) with the additive noise $\Omega(\phi)$, we get

Fig. 1. (Color online) The schematic diagram of two delayed SL system. LD, EC1, EC2 and RTO denotes the laser diod, first external cavity, second external cavity and real-time oscilloscope respectively.
\[
\frac{dE}{dt} = 0.5E(n-1) + \eta_1 E(t-\tau_1)\cos(\phi - \phi(t-\tau_1) + \omega\tau_1) \\
+ \eta_2 E(t-\tau_2)\cos(\phi - \phi(t-\tau_2) + \omega\tau_2) + K\Omega(\beta), \quad \frac{d\phi}{dt} \\
= 0.5\alpha(n-1) - \eta_1 \frac{E(t-\tau)}{E}\sin(\phi - \phi(t-\tau) + \omega\tau) \\
- \eta_2 \frac{E(t-\tau)}{E}\sin(\phi - \phi(t-\tau_2) + \omega\tau_2), \quad \frac{dn}{dt} \\
= \frac{2}{T}(\rho - 0.5(n-1) + mE^2)
\]

where $K$ is the noise strength. For the case of induced noise, we choose $\Omega(\beta) = \frac{1}{\beta^2}$, with $\beta = 0.5$.

2.2. Existence of chaos-bifurcation and 0–1 test

In this section, we have investigated the dynamics of system (2) and (3) with the changing feedback rates $\eta_1, \eta_2$. The analysis is based on single and two parameters bifurcation analysis followed by 0–1 test for chaos. These results are also verified by measuring asymptotic characteristics of the phase space corresponding to the systems (2) and (3).

In order to measure the bifurcation of the system (2), we have calculated local maxima $maxE$ for each $\eta_1, \eta_2 \in [0.01, 1]$. The corresponding diagrams are shown in Fig. 2a and b respectively. It can be observed from Fig. 2 that system (2) reveals a consistent region of multiperiodicity when $\eta_1 > 0.08$ for fixed $\eta_2 = 0.01$. Moreover, an increasing trend of $maxE$ is being observed under the same conditions. On the other hand, a consistent region of multiple period starts for $\eta_2 > 0.05$ with $\eta_1 = 0.01$. The increasing trend of $maxE$ is also observed in this case. In both the cases, the system (2) reveals increasing as well as decreasing periodic structures in the corresponding bifurcation diagrams (see Fig. 2a and b). By changing the parameters $\eta_1$ and $\eta_2$, it can be observed that the output power can be modulated without any effect of noise. In that case, placement of the external cavity plays an important role to modulate the output power. We have also observed the bifurcation phenomena of the system (3) with $\Omega(\beta) = \frac{1}{\beta^2}$, $\beta = 0.5$.

For both $\eta_1$ (with fixed $\eta_2 = 0.01$) and $\eta_2$ (with fixed $\eta_1 = 0.01$) the corresponding diagrams are given in Fig. 2e and f respectively. From the figures, it can be seen that the system (3) possesses multiperiodicity when $\eta_1, \eta_2 > 0.03$. However, many single strips are found when $\eta_1, \eta_2 < 0.03$. It is also interesting to observe that, under the influence of noise, some periodic windows corresponding to Fig. 2 a, b are changed to the chaotic state in Fig. 2e and f respectively. A brief investigations on the changes in the dynamics and the corresponding phase spaces without and with the noise is given in Fig. 4.

To measure the composite effect of $(\eta_1, \eta_2)$ on the system (2) and (3), we have computed bifurcation over the region $[0.01, 0.05] \times [0.01, 0.05]$. The corresponding two parameter bifurcation diagrams are shown in Fig. 3a and b. Fig. 3a shows some regions with specific colors-red, yellow, green, blue and purple respectively. It indicates the occurrence of single, double and multiple periods in the phase space of the respective dynamics. As these appear, it signifies that the dynamics become regular as well as chaotic with the variable $(\eta_1, \eta_2)$. It can be also observed in Fig. 3a that the system (2) has no single/double periodicity or so-called simple structure for $(\eta_1, \eta_2) > (0.045, 0.045)$. We can call this limit as ‘point of no simplicity’. In fact, the point of no simplicity indicates the beginning of continuous chaotic region for the system (2). On the other hand, Fig. 3b shows multiple periods over the region $[0.01, 0.05] \times [0.01, 0.05]$. In fact, by observing the corresponding color bar (in Fig. 3b), it can be easily verified that the system always produces a large amount of periods with the variations of $(\eta_1, \eta_2) \in [0.01, 0.05] \times [0.01, 0.05]$. The bifurcation diagrams (shown in Fig. 3a, b, e, f and Fig. 3a and b) indicate period route to chaos phenomena in the SL system. In fact it recognizes the region of $\eta_1, \eta_2$ for which SL possesses multiple periods. The two parameter bifurcation analysis confirms that the dynamics can be made more complex by changing the composite feedback rates $(\eta_1, \eta_2)$ and the noise strength $K$. However, bifurcation analysis cannot recognize chaos from the multi-periodicity of a system and thus we have further investigated chaotic phenomena of the SL system with and without noise by 0–1 test [45,46]. The 0–1 test, proposed by Georg et al., is based on statistical theory of time series. The main advantage of this method is it does not
require any time-delay and embedding dimension of the time series. In this method, two translation variables from a time series \(x(j), j = 1, 2, \ldots, N\) are constructed by

\[
p_c(n) = \sum_{j=1}^{n} x(j) \cos(jc), \quad q_c(n) = \sum_{j=1}^{n} x(j) \sin(jc),
\]

where \(c \in (0, x)\) and \(n = 1, 2, \ldots, N\).

The diffusive (or non-diffusive) behavior of \(p_c\) and \(q_c\) is then investigated by measuring the mean square dispersion \(M_c\), where

\[
M_c = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} [p_c(j + n) - p_c(j)]^2 + [q_c(j + n) - q_c(j)]^2,
\]

(5)

where \(nN\). The limiting value of \(M_c\) is assured only for \(n \leq n_{\text{cut}}\), where \(n_{\text{cut}} = \frac{m}{10}\) reveals good result. In \(\beta\), \(\eta\), the theory assures that a dynamics is regular if and only if \(M_c\) is bounded function in time. It can be also observed that the dynamics will be chaotic if and only if \(M_c\) shows linear scaling behavior in time. In order to investigate the behavior of \(M_c\), the asymptotic growth of \(MC\) is calculated. The asymptotic growth of \(M_c\) is given by

\[
K_c = \lim_{n \to \infty} \frac{\log M_c(n)}{\log n}.
\]

(6)

The value of \(K_c\) close to 1 and 0 indicates chaotic and regular dynamics respectively. We have investigated the fluctuation of \(K_c\) for the SL system with variable feedback rates. The investigation is done in both noisy and noise free conditions. The fluctuations of \(K_c\) for the variable \(\eta_1\) and \(\eta_2\) are shown in Fig. 2c and d respectively. The same under noisy condition are shown in Fig. 2g and h respectively. From the Fig. 2c, d, g and h, it can be observed that the \(K_c\) values are strongly correlated with the bifurcation diagrams (Fig. 2a, b and e, f). The 0–1 test indicates that chaos appears in SL for \(\eta_1 = 0.055\) and also for \(\eta_1 \in (0.059, 0.068), (0.072, 0.1)\) with few exceptions for a fixed \(\eta_2 = 0.01\), while for fixed \(\eta_1 = 0.01\), chaos appears for \(\eta_2 \in (0.055, 0.098)\) in noise free states. However, in noise perturbed conditions, the fluctuation of \(K_c\) in Fig. 2g and h, indicates that the chaos appears in \(\eta_1 \in [0.01, 0.1]\), for a fixed \(\eta_2 = 0.01\) and also in \(\eta_2 \in [0.01, 0.1]\) for a fixed \(\eta_1 = 0.01\). We further investigated two parameter 0–1 test with the variable \((\eta_1, \eta_2)\) in both noisy and noise free conditions. The corresponding contours are shown in Fig. 3c and d respectively. It can be observed from Fig. 3c that \(K_c\) is always close to 1 for \((\eta_1, \eta_2) > (0.044, 0.015)\), which signifies the beginning of chaotic regime. On the other hand, from Fig. 3d it can be observed that chaos appears for \((\eta_1, \eta_2) > (0.01, 0.05)\) as \(K_c \approx 1\) there. Both the results correspond to the results in 2D bifurcation diagram given in Fig. 3a and b. The corresponding complex movements can be observed in their phase space trajectory. So, we analyze the asymptotic behavior of the system (2) and (3) by observing their corresponding attractors given in Fig. 4. Fig. 4b–d indicates that chaotic phenomena can be observed in both (2) and (3) with the feedback rates \((\eta_1, \eta_2) = (0.15, 0.15), (0.15, 0.34), (0.34, 0.15)\) and \(\beta = 0.5\). At \(\eta_1 = 0.05, \eta_2 = 0.05\), system (2) possesses non-chaotic attractors. However, it becomes chaotic under the noise with strength \(K = 0.1, 0.3\). In fact, denser orbits having complex movements in the trajectories can be observed in the attractors with the increasing noise strength. The same trend has been observed in the remaining cases. So, the analysis of attractors indicates that the changes of \(\eta_1, \eta_2\) and \(K\) must affect the dynamics as well as complexity of the systems (2) and (3). From the figures, we have further investigated the range of the output in each phase spaces under various \(K\) and \(\eta_1, \eta_2\). It has been observed that range of \(E\) increases as \(K\) increases. It indicates strong correlation between \(E\) and \(K\). To understand this under the effect of \(\eta_1, \eta_2\) in noise free conditions, we have investigated the range of \(E\) in Fig. 4a–d.

In order to quantify the laser intensity and complexity of the systems (2) and (3), we have introduced a window based approach which is given in the following section:

3. Window output power and complexity

The output power (OP) modulation of the electric field
\( E = \{E_i, E_2, \ldots, E_n\} \) can be derived by measuring \( \| E \| \). For LK-system, \( \| E \| \) can be derived by
\[
\| E \| = \sqrt{\frac{1}{N} \sum_{i=1}^{N} E_i^2},
\]
(7)
where \( N \) is the length of the solution component \( E \).

We define an average measure of \( E \) by windowing technique as follows:

### 3.1. Output power

To define the window output power (OP) from \( E = \{E_i, E_2, \ldots, E_n\} \), we first consider the windows \( W_{s,L} = [1 + sL, 1 + (s + 1)L] \), where \( s = 0, 1, 2, \ldots, n \) and \( (s_0 + 1)L \leq N \). Then, for each fixed \( L \), we can partitioned \( E \) by \( P_s \) by
\[
P_s = \{E_{s+L}, E_{s+L+1}, \ldots, E_{n}\},
\]
(8)
where \( L \) is so chosen that \( 1 + (s + 1)L \leq N \).

From the above definition, it can be easily verified that the values of OP are different for variable partitions \( P_s \). Thus, it can reveal different trends of OP over the various window. In order to get invariant OP, we define a statistical average of all OPs over variable partitioned \( P_s \).

The mean or average of OP over each variable partition \( P_s \) defined by
\[
E = \langle \sum_{i=1}^{N} E_i^2 \rangle,
\]
(9)
where \( \langle \cdot \rangle \) denotes the statistical mean of \( \cdot \).

We next define window sample entropy (sampEn) using the concept of recurrence plot.

### 3.2. Optical complexity - a window based entropy

Consider an \( n \)-dimensional phase space \( P \) of \( m \)-dimensional system, is given by \( P = \{X_i; X_i \in R^n, i = 1, 2, \ldots, L\} \). Then, the corresponding recurrence plot (RP) can be defined by a matrix \( (R_{ij})_{k \times l} \), where
\[
R_{ij} = \Theta(\epsilon - \| X_i - X_j \|) \quad (\Theta \text{and } \epsilon \text{ represents Heaviside function and recurrence threshold})
\]
respectively. The outcomes \( '1' \) and \( '0' \) indicate ‘recurrence’ and ‘non-recurrence’ respectively. The number of points recurs to \( X_i \)-th point can be calculated by
\[
\mu_i^{n}(\epsilon) = \sum_{j=1}^{L} \Theta(\epsilon - \| X_i - X_j \|).
\]
(10)
For any \( Y_j \in R^{n+1} \) in the \( (n + 1) \)-dimensional phase space, the number of recurrent point is therefore calculated by
\[
\nu_i^{n+1}(\epsilon) = \sum_{j=1}^{L} \Theta(\epsilon - \| X_i - Y_j \|).
\]
(11)
Since number of recurrence over the phase space indicates probability of the recurrence, we can calculate \( p^{n+1}(i, \epsilon) \) and \( p^{n+1}(i, \epsilon) \) by
\[
p^{n+1}(i, \epsilon) = \frac{1}{L} \nu_i^{n+1}(\epsilon), \quad \mu_i^{n}(\epsilon) = \frac{1}{L} \nu_i^{n+1}(\epsilon).
\]
(12)
Thus sample entropy \( S \) of the whole phase space is defined as
\[
S(\epsilon) = -\log \left( \frac{\sum_{i=1}^{L} \mu_i^{n}(i, \epsilon)}{\sum_{i=1}^{L} \mu_i^{n+1}(i, \epsilon)} \right).
\]
(13)
The value of \( \epsilon \) taken as 0.2×standard deviation (data) in standard sampEn method. For the recurrence based SampEn approach, we fixed \( \epsilon \) as 0.1×diameter of the phase space.

Fig. 4. (Color online) (a)-(d) represent the phase spaces of the system (2) with \((\theta_1, \theta_2) = (0.05, 0.05), (0.15, 0.15), (0.05, 0.34)\) and \((0.34, 0.05)\) respectively. (e)-(h) and (i)-(l) indicate the phase spaces of (3) with similar respective values of \((\theta_1, \theta_2)\) for noise strength \(K = 0.1\) and \(0.3\) respectively. In each case, numerical simulation is done with time \(t = 10,000\). The values of \( \omega = 0.1, p, T, n \) are taken as 0.1, 5, 2, 100, 6 respectively.
We next partition the phase space using window $W_k$ of length $'L_0'$ defined by

$$W_k = (x_{k}^{L_0+1}(1), x_{k}^{L_0+2}(2), \ldots, x_{k}^{L_0+n}(n)), \quad (14)$$

where $x_{k}^{L_0+n}(J)$ column vector of J-th component of n-dimensional state vector for $k = 1, 2, \ldots, K_0$ so that $K_0 + 2L_0 \leq L$.

We now implement the concept of SampEn (given in (10)) to construct a window based complexity measure.

For $i$th window of the phase space, we define entropy $S_i$ as $k = 1, 2, 3,\ldots, K_0$ as

$$S_i = -\log \left( \frac{\sum_{j=1}^{K_0} P_{W_i}^{(n)}(i, e)}{\sum_{j=1}^{K_0} P_{W_i}^{(n)}(i, 0)} \right), \quad (15)$$

where $P_{W_i}^{(n)}(i, e)$ denotes probabilities of $i$-th recurrent point on the trajectory for the n-dimensional window phase spaces $W_i$, $k = 1, 2, 3,\ldots, K_0$.

Then average or mean window complexity (S) is defined as

$$S = \frac{1}{K_0} \sum_{i=1}^{K_0} S_i. \quad (16)$$

We call it Optical complexity (OC) as we have used it to measure the complexity of the SL dynamics under various optical feedback. In the following section, the variation of OP and OC has been discussed under various optical feedback rates and noise strengths. We have also investigated the correlation between the OP and OC.

### 3.3. Analysis of output power and complexity for the semiconductor laser

#### 3.3.1. Noise free SL

The OP and OC are computed with various $0.15 \leq \eta_1, \eta_2 \leq 0.35$, using (9) and (16) respectively. The corresponding graphs are shown in Fig. 5a, b, d and e. The figures show that by using standard and window method, increasing trend can be observed in both $E$ and $S$ respectively. In fact, the values of $E$ are always higher in window methods. It implies the effectiveness of the window method to measure the output power of the system (2). To measure the coherency between $E$ and $S$, we first observed their oscillations for both $0.15 \leq \eta_1, \eta_2 \leq 0.35$. It can be observed from the Fig. 5a, b, d and e that the fluctuating pattern between $S$ and $E$ are almost similar by window method. On the other hand, there exist some mismatches between the same in standard method [33]. It indicates weak coherency between $E$ and $S$. In order to confirm these, we have measured their correlations. Fig. 5c and f shows the corresponding cross-correlation diagrams. From the figures, it can be observed that correlations between $S$ and $E$ are always higher in window method compared to the standard method. It follows from the above analysis that an optimized OP can be generated by the window method which are strongly correlated with OC.

Next, we have investigated the composite effect of $(\eta_1, \eta_2)$ on $E$ and $S$. The corresponding contours are shown in Fig. 6a, b, d and e. It can be observed from the Fig. 6a and d that $E$ has an increasing trend over the region $[0.15, 0.3] \times [0.15, 0.3]$. It indicates that the output power of (2) can be enhanced by the increasing composite feedback rates $(\eta_1, \eta_2)$ in $[0.15, 0.3] \times [0.15, 0.3]$. However, larger $E$ can be gained with a faster rate by the window method (see the Fig. 6a and d). It establishes the effectiveness of our introduced method. The similar trends are also observed in Fig. 6b and e. It signifies the increasing complexity of the system (2). To quantify the similarity between the pattern of $E$ and $S$, we have computed the correlation contours over the region $[0.15, 0.3] \times [0.15, 0.3]$. The corresponding diagrams are shown in Fig. 6c and f. From the figures, it can be seen that the values of correlations in Fig. 6f are always higher than the same in Fig. 6c. It indicates a strong correlation between $S$ and $E$. Thus the analysis of $(\eta_1, \eta_2)$ also proves the robustness of the window method.

We have further investigated the variation of OP and OC of the systems (2) under variable $(\tau_1, \tau_2) \in [20, 42] \times [30, 62]$. The results are shown in Fig. 7a and b. It can be observed from Fig. 7 a and b that, changes of $(\eta_1, \tau_2)$ affect the output power and optical complexity of the
systems (2). In fact, \( \tau_2 \) takes an important role to increases the OP and OC. However, the changes in OP and OC are not homogeneous. So, monotonic patterns have not been found in both the corresponding contours. It can be also observed that increasing and decreasing patterns are similar in both OP and OC. To verify the similarity between OP and OC, we have measure the cross-correlations between OP and OC under variable \( \tau \). The corresponding cross-correlation surface is given in Fig. 7c, which shows high correlation at the cross-correlation lag \( (0,0) \). It signifies high correlation between OP and OC under variable \( (\tau_1, \tau_2) \).

From the whole study in noise free LK-system, we can therefore state that an optimized output power \( E \) can be generated by window method, which is strongly correlated with OC.

3.3.2. Noise induced SL

The noise effect has been individually investigated by measuring the
increment of \( E \) and \( S \) under various \((\eta_1, K)\), \((\eta_2, K)\) \(\in [0.1, 0.3] \times [0.1, 0.3] \). To compute these, we have fixed \( \eta_1 = 0.15 \) and \( \eta_2 = 0.15 \) in respective cases. The investigation is done for both standard and window methods.

For the standard method, OP and OC are described in Fig. 8a, b, d and e. From Fig. 8a and d, the increasing trend of \( E \) can be observed in \([0.1, 0.3] \times [0.1, 0.3] \). It indicates an enhancement of the OP under composite variation of \((\eta_1, K)\) and \((\eta_2, K)\) respectively. To observe the effect of same in OC, we have observed the nature of \( S \) in Fig. 8b and e. No increasing pattern has been found in the figures. In fact, the pattern shows indefinite trend. It signifies an unpredictable complexity under various \((\eta_1, K)\) and \((\eta_2, K)\). This unpredictable complex phenomenon also indicates weak correlation between \( E \) and \( S \). To quantify this, we have computed cross-correlation between \( E \) and \( S \) over the region \([0.1, 0.3] \times [0.1, 0.3] \). The corresponding cross-correlation surfaces are shown in Fig. 8c and f. It can be observed from the both figures that the value of correlation is near to zero at (0,0). It implies weak correlation between \( S \) and \( E \). So, the analysis of OP and OC by standard method indicates that larger output can be gained with unpredictable OC. It contradicts with the asymptotic analysis of the dynamics of the SL under noisy condition.

We have done the same analysis of OP and OC with window method. The corresponding contours are given in Fig. 9a–d. As in the previous analysis, we also observe the increasing pattern of the output power. However, faster increment of \( E \) can be seen in this case (see Fig. 9a and c). It implies larger output power with small changes of \((\eta_1, K)\) and \((\eta_2, K)\) respectively or in other words, the output power can be enhanced by window methods under small changes of \( \eta_2 \) and \( K \). The similar increasing trend can be observed in Fig. 9b and d. Both the figures show increasing complexity of the system (3) under various \((\eta_1, K)\), \((\eta_2, K)\) \(\in [0.15, 0.34] \times [0.1, 0.3] \). It can be also verified that the increment of complexity \( S \) is very fast in window-based method under very small changes of \((\eta_1, K)\), \((\eta_2, K)\). It confirms the existence of correlation between \( S \) and \( E \). The quantification is done by computing normalized cross-correlation between \( E \) and \( S \) for both the variables \((\eta_1, K)\) and \((\eta_2, K)\). The corresponding cross-correlation surfaces are given in Fig. 10. From the figures, it can be observed that the maximum peaks (cross-corr=0.8) occurred at (0,0). It confirms that a strong correlation exists between \( S \) and \( E \) for both of \((\eta_1, K)\) and \((\eta_2, K)\). So, the whole analysis reveals an indication of increasing complexity, as well as the output power of the noise perturbed two delayed SL-system.

4. Conclusion

In this paper, our main intention is to establish the relation between output power and the instabilities of time delayed SL. The dynamics and the intensity of an SL model have been investigated under both noisy and noise-free conditions. In the case of noisy environment, the effect of power noise with optimal intensity has been investigated. The system dynamics has been quantified by the correlation between its attractor’s pattern and bifurcation analysis. This result indicates that attractors become denser when the values of the feedback rates increase within certain intervals. The instability and the inherent information of the system can be quantified by the measure of the complexity using SampEn. In this article, an entropy measure: optical complexity, which is a recurrence based SampEn, is defined by the concept of recurrence plot and window phase space technique. The analysis of OC confirms the increasing complexity and output power with the increment of the feedback rates. The result shows a strong correlation between OP and OC under both noisy and noise free conditions. The same analysis has been done for the composite effect of feedback rates and noise strength. Under a certain range of the feedback rates and noise strength, we have found the aforesaid correlation with high degree. On the other hand, the usual approach does not show any significant correlations between output power and the optical complexity under different parameters. Thus, our analysis
confirms the SL model can be generated high intensity with increasing feedback rates as well as induced noise strengths. The optical complexity also established a high correlation with the output power. Thus an SL model with high optical complexity can produce higher output power. The results are effective to quantify the dynamics and complexity of a chaotic SL.

References
