The Iris billiard: Critical geometries for global chaos

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ABSTRACT

We introduce the Iris billiard that consists of a point particle enclosed by a unit circle around a central scattering ellipse of fixed elongation (defined as the ratio of the semi-major to the semi-minor axes). When the ellipse degenerates to a circle, the system is integrable; otherwise, it displays mixed dynamics. Poincaré sections are presented for different elongations. Recurrence plots are then applied to the long-term chaotic dynamics of trajectories launched from the unstable period-2 orbit along the semi-major axis, i.e., one that initially alternately collides with the ellipse and the circle. We obtain numerical evidence of a set of critical elongations at which the system undergoes a transition to global chaos. The transition is characterized by an endogenous escape event, $\mathcal{E}$, which is the first time a trajectory launched from the unstable period-2 orbit misses the ellipse. The angle of escape, $\theta_{\text{esc}}$, and the distance of the closest approach, $d_{\text{min}}$, of the escape event are studied and shown to be exquisitely sensitive to the elongation. The survival probability that $\mathcal{E}$ has not occurred after $n$ collisions is shown to follow an exponential distribution.

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The Iris billiard, a unit circle enclosing a central scattering ellipse, of fixed elongation, is a generic system, i.e., one that exhibits mixed dynamics. The chaotic dynamics within this system display incompletely understood features, such as stickiness. Transport barriers within the phase space, which play a fundamental role in the system's mixed dynamics, may be created or destroyed by varying the ellipse elongation. We explore the consequences of such a variation on the long-term evolution of chaotic trajectories using recurrence plots. These display features identify a set of critical elongations that demark a transition to global chaos. The observable properties of the dynamical transition are characterized.

I. INTRODUCTION

Billiards1 are Hamiltonian systems in which a point particle moves freely within a compact, Euclidean domain. The particle undergoes elastic collisions at the domain's edge and so exhibits dynamics that are purely determined by the interplay between its initial conditions and the confining boundary. These systems exhibit three behaviors: (1) Regular (i.e., periodic or quasiperiodic orbits, as found in circular,2 elliptic,4 or confocal elliptic4 billiards), (2) Ergodic, with orbits that fill the entire phase space, as found in the Sinai stadium,2 the Bunimovich stadium,1 and cardioid1 billiards. (3) Mixed dynamics, i.e., with coexisting regular and irregular trajectories, such as found in the family of limaçon, eccentric annular, and mushroom billiards.5,7,8

These systems, which have been the focus of mathematical research,2,10–13 provide insight into physical phenomena such as encountered in celestial mechanics,12,15 statistical mechanics,9,16,17 tokamak physics,18,19 as well as being important models for quantum chaos.20–22 Billiard systems connect experimental physics and mathematics through experiments employing both two and three dimensional geometries that may be either open or closed. Examples include situations where particles or waves are confined to cavities or other homogeneous regions such as wave guides,10 electrons in semiconductors confined by electric potentials,15 and atoms
interacting with laser beams.\textsuperscript{25,26} Dynamical tunneling between classically isolated phase space regions has also been investigated and observed in both desymmetrized mushroom and eccentric annular superconducting microwave resonators.\textsuperscript{27,28} Escape rates of open billiards are a characteristic that is both experimentally accessible\textsuperscript{29} and important for transport properties of many related systems such as fractal conductance fluctuations.\textsuperscript{30}

Mixed dynamics have many interesting and unexpected characteristics, such as the existence of dynamical barriers to chaotic transport\textsuperscript{15} and quasi-regular chaotic motion near regions of stability, known as "stickiness,"\textsuperscript{32–35} of which there are two types.\textsuperscript{35} “Internal” stickiness presents in systems with no islands of stability, such as the Bunimovich stadium or else is due to the presence of marginally unstable periodic orbits (MUPOs) that are completely contained within the chaotic sea, such as in mushroom billiards.\textsuperscript{36} “External” stickiness, however, arises due to the existence of the boundaries between regular and chaotic regions. Stickiness results in non-exponential decays of both the time-correlation functions and Poincaré recurrence distributions of the system’s chaotic dynamics.\textsuperscript{37–39}

To understand these behaviors, recurrences,\textsuperscript{41,42} which are events characterized by a given trajectory occupying a state close to one already visited, are often considered. Recurrence plots (RPs)\textsuperscript{33,34} permit the qualitative and quantitative study of this property. This tool has already been applied in the fields of economy,\textsuperscript{43} physiology,\textsuperscript{44} ecology,\textsuperscript{45} neuroscience,\textsuperscript{46} and astrophysics.\textsuperscript{47} Although recurrence statistics have been extensively studied in billiards,\textsuperscript{31,43,45} RPs appear to have only been applied to position recurrences in a two particle billiard system\textsuperscript{50} and, more recently, to the Poincaré section of an eccentric annular billiard.\textsuperscript{51} In this paper, time recurrences in the Poincaré section will be considered.

Section II fully describes the model. Section III presents a set of Poincaré sections for different ellipses and details their features. Section IV applies RPs and a new time measure to chaotic trajectories starting with the unstable period-2 orbit for many ellipse parameters. Section V identifies and further studies the endogenous escape event. Section VI discusses the results and concludes.

II. THE SYSTEM

The billiard domain, $B \subset R^2$, has a two-part, continuous, boundary, $\partial B = \bigcup \partial B_i$, where each $\partial B_i$ is piece-wise smooth. The initial conditions of a given trajectory are defined by the arc length distance, $s = \theta \in (-\pi, \pi]$, along the outer circular boundary, and the initial direction of motion described by the angle between the initial velocity and the center-facing normal to the outer boundary, $\beta \in (-\pi/2, \pi/2]$. See Fig. 1. At any instant, the point particle, mass $m=1$, is described by its position, $q \in B$, and momentum, $|p| = 1$. The dynamics is governed by the Hamiltonian,

$$H(q, p) = \begin{cases} \frac{p^2}{2}, & q \in B/\partial B, \\ \infty, & q \in \partial B. \end{cases}$$

(1)

The infinite boundary potential causes every collision to be elastic. Therefore, the component of the momentum projected onto the normal at the point of each collision changes sign, while the tangential component stays constant. The momentum vector, $\mathbf{p}_{i+1}$, after the $i$th collision at point $q_i$, is

$$\mathbf{p}_{i+1} = \mathbf{p}_i - 2[\mathbf{p}_i \cdot \hat{\mathbf{n}}(q_i)]\hat{\mathbf{n}}(q_i).$$

(2)
FIG. 2. A selection of regular, circle map, trajectories. (a) A periodic orbit with frequency ratio \( \omega_1/\omega_2 = 1/3 \). (b) A periodic orbit with frequency ratio \( \omega_1/\omega_2 = 2/7 \). (c) A quasiperiodic orbit with frequency ratio \( \omega_1/\omega_2 = 1/\sqrt{2} \) that would densely fill the annulus if allowed to run for infinite time. All trajectories were generated with \( 10^4 \) collisions.

FIG. 3. A selection of orbits that collide with the ellipse. Geometry: \( a = 0.2, e = 4 \). (a) Periodic orbit: initial conditions: \( \theta = 2.19, \beta = 0.577 \). (b) Quasiperiodic orbit: initial conditions: \( \theta = 2.48, \beta = 0.577 \). (c) Chaotic orbit: initial conditions: \( \theta = 3, \beta = 0.577 \). All trajectories were generated with \( 10^4 \) collisions.
\( \hat{n}(q) \) is the normal of the boundary at each collision. The equation of the ellipse is
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{3}
\]
where \( a \) and \( b \) are the semi-major and minor axes, \((b \in (0, 1) \) and \( a \in (0, b])\). We use the elongation, \( e = b/a \), as a measure of symmetry breaking. When \( e = 1 \), the system reduces to the centered annulus, which is totally symmetric, and, therefore, integrable by conservation of angular momentum.

All possible trajectories fall into one of two mutually exclusive categories: those that never hit the central ellipse and those that do. The first type, illustrated by Figs. 2, is characterized for all quasiperiodic orbits by an angle \( \beta \) that obeys the following tangency condition:
\[
\beta \geq \beta_c = \arcsin b, \tag{4}
\]
where \( b \) is the radius of the inaccessible circular region, defined by a caustic edge, as seen in Fig. 2(c). The trajectories are regular (i.e., either periodic or quasiperiodic, with constant \( \beta \)) and are given by the unperturbed circle map,
\[
\theta_{n+1} = \theta_n + \Omega \mod 2\pi, \text{ with } \Omega = \pi - 2\beta. \tag{5}
\]
If \( \Omega/2\pi \) is rational, the motion may be represented as a quotient of coprime numbers such that \( \Omega = 2\pi \omega_1/\omega_2 \), with \( \omega_1, \omega_2 \in \mathbb{Z}^+ \). In this case, the orbit closes on itself after a finite number of iterations; i.e., it is periodic and will form polygons as in Figs. 2(a) and 2(b). If \( \Omega/2\pi \) is irrational, the sequence \( \{\theta_n\} \) ergodically fills \([0, 2\pi]\), as time tends to infinity, as shown in Fig. 2(c). It is well known that rational approximations of some irrational frequencies are obtained through its continued fraction representation.5

Numbers can be irrational to different degrees, the highest of which is the golden mean, represented by a continued fraction whose partial quotients are all equal to 1 and is an element of the set of noble frequencies whose partial quotients always end in ones, although the first quotients may be different. The transport barriers defined by these frequencies are more robust under perturbation, as has been verified in a work that applied Greene’s residue criterion to numerically determine the frequency of the final destroyed invariant transport barrier for both the standard map and the double pendulum.5

If the tangency condition is not satisfied, some rational \( \Omega = \omega_1/\omega_2 \) orbits may continue to exist over a finite range of ellipse elongations. These trajectories form \( \omega_2 \)-polygons whose radius depends on the intersection of the enclosed caustic,
\[
r_{\omega_1, \omega_2} = \left| \cos \left( \frac{\omega_1 \pi}{\omega_2} \right) \right|. \tag{6}
\]
The minimum value of \( e \) at which the rational polygon may exist at some orientation without intersecting the ellipse is
\[
e_{\text{min}} = \left[ \frac{1}{\cos^2(\pi/\kappa(\omega_2))} \left( \frac{r_{\omega_1, \omega_2}^2}{\omega_2^2} - 1 \right) + 1 \right]^{1/2}, \omega_2 \in \{3, 4, 5 \ldots \}, \tag{7}
\]
where
\[
\kappa(\omega_2) = 1 + \omega_2 \mod 2. \tag{8}
\]
See Appendix A for the derivation. Fixing \( \omega_1 \) and taking the limit \( \omega_2 \rightarrow \infty \) returns the limit,
\[
\lim_{\omega_2 \rightarrow \infty} e_{\text{min}} = 1/a. \tag{9}
\]
On the contrary, the maximum value of \( e \) at which the \( \omega_2 \)-polygon intersects the ellipse for any orientation is
\[
e_{\text{max}} = r_{\omega_1, \omega_2}/a \tag{10}
\]
and so approaches the same limit as \( e_{\text{min}} \) when \( \omega_2 \rightarrow \infty \).

The second type of trajectory, which involves collisions with the ellipse, illustrated by Fig. 3, results in rotational and librational periodic and quasiperiodic orbits (i.e., \( \beta \neq \text{constant} \)) as well as chaotic orbits. Appendix B includes a stability analysis for the simplest period-2 orbits along the axes of symmetry of the ellipse. The orbit along the semi-minor axis is stable for all elongations. In contrast, the orbit along the semi-major axis is unstable for all elongations greater than one. However, the range of dynamics in Fig. 3 can only be completely visualized by considering the phase space.

III. POINCARÉ SECTIONS

The phase space is a set of points that fully describe the system. For planar billiard systems, it is four dimensional, \((x, y, v_x, v_y)\). However, the complete description of an orbit can be shown using a two-dimensional Poincaré section, via Birkhoff coordinates, \((\theta, \sin(\beta))\). \(\theta \) is the arc length of the outer circle collision. Due to the system’s axial symmetry, \(\theta \) can be restricted to the range \(\theta \in (-\pi/2, \pi/2)\). \(\sin \beta \in (-1, 1)\) is the momentum component, at the point of collision, tangential to the boundary. The periodicity in \(\theta \) makes the Poincaré section topologically equivalent to an annulus, \(\mathbb{S} \times [-1, 1]\).

Figure 4 shows a series of sections constructed by trajectories launched from initial conditions spanning \(\theta, \beta \in (-\pi/2, \pi/2)\). If the tangency condition, Eq. (4), is obeyed, trajectories follow the rotational curves above and below the central mixed region, defined by \(\beta = \text{constant} \). Lines corresponding to periodic, but from a disjoint set of initial conditions, and rotational quasiperiodic motions shown in Fig. 2 smoothly wind around the annulus. These circles are homotopically non-trivial and are known as “rotational circles.” Increasing the elongation of the ellipse, \(e\), until the tangency condition, Eq. (4), is just violated, causes the trajectory to make contact with the ellipse, therefore deforming the orbit’s associated invariant rotational circle in the Poincaré section. The curves often persist with increasing elongation, as predicted by Kolmogorov–Arnold–Moser (KAM) theory, and there is, consequently, no flux of trajectories between the regions partitioned by the deformed circle. As the elongation is further increased, however, the invariant curves will be increasingly deformed until some critical value is reached (each corresponding to the quasiperiodic frequency ratio of the deformed curve in question), at which point the curve is destroyed. When \(\beta < \beta_c\), the Poincaré section displays mixed dynamics; i.e., it is divided into several invariant components.
The center of every Poincaré section presented, i.e., \( \theta = 0 \), \( \beta = 0 \), or equivalently \( \pm \pi \), is an elliptical fixed point for all geometries corresponding to the stable period-2 orbit along the semi-minor axis. Sets of concentric, homotopically trivial circles, representing librational quasiperiodic motion [as seen, for example, in Fig. 3(b)] surround the stable period-2 orbit for all geometries, as well as from other geometry-dependent elliptical periodic orbits. It is thought that these curves only cause a limited impediment to the diffusion of chaotic orbits since they do not encircle the entire annulus.\(^5\)\(^3\) Librational circles occur in concentric sets. The outermost forms the critical boundary of the island of stability, which can be destroyed by an arbitrarily small perturbation.

The hyperbolic period-2 orbit along the ellipse semi-major axis is always within the chaotic sea. (The exception to this is when \( e = 1, \) i.e., when the system is integrable.) Boundaries between the regular and chaotic components of the phase space are often characterized by scale invariant structures. Such features are illustrated by Fig. 5, in which an island archipelago is magnified, indicating where a critical curve used to lie. Insets 2–4 clearly show scale invariant structures that reveal different dynamics present inside the islands, i.e., quasiperiodic trajectories enclosing narrow stochastic layers. These island chains create partial barriers to chaotic transport and are the source of external stickiness.

**IV. RECURRENCE PLOTS AND QUANTIFICATION ANALYSIS**

The Poincaré recurrence theorem states that for a finite measure preserving transformation, almost every point in a finite measure set will return to its neighborhood infinitely many times. Therefore, even though very close chaotic trajectories exponentially deviate in finite time,\(^5\)\(^6\) they must, eventually, return arbitrarily close to their initial conditions and evolve in ways similar to before.\(^7\) Although this theorem gives no indication of the frequency at which recurrences occur, RPs allow the quantitative and qualitative study of this feature.\(^4\)\(^4\) During an orbit of \( N_{\text{coll}} \) total collisions, there are \( N_{\text{col}} \) collisions with the outer boundary. The time evolution of the points corresponding to a trajectory in the Poincaré section can be labeled as \( \{v_i\}_{i=1,\ldots,N_{\text{col}}} \). A state, \( v_i \), is defined as recurrent to a former state, \( v_j \), if sufficiently close.

The \( L_\infty \) norm is used to define the neighborhood around each point, which defines a square of length \( \epsilon \) with \( v_j \) at its center such that \( v_i \) is a recurrent state of \( v_j \) if and only if the two states lie within the same square region. The value of \( \epsilon \) is important. If too small, no recurrences would be recorded in a finite time. If too big, every point would be recorded as a recurrence of every other point, leading to artifacts unrelated to the dynamics. Although these artifacts...
Chaos may be analytically determined for simple periodic and quasiperiodic motions, \(^*\) in general, they cannot be completely removed by any known means. We follow the usual convention of defining \( \epsilon \) as 10% of the width of its corresponding phase component. \(^1\)

The binary, \( N_i \times N_j \), recurrence matrix is defined as

\[
R_{ij} = \Theta(\epsilon - ||v_i - v_j||), \quad i, j = 1, \ldots, N,
\]

where \( \Theta(\cdot) \) is the Heaviside function.

RPs are the graphical representation of \( R_{ij} \). The value “1,” encoded by a black point, indicates that \( |v_i - v_j| < \epsilon \). Otherwise, points are blank, representing the value “0.” All RPs will show a diagonal line, i.e., \( R_{ij} = 1 \ \forall i = j \), known as the Line of Identity (LOI). RPs display many patterns associated with different behaviors. At the small scale, they exhibit isolated points, diagonal lines, and vertical lines (the combination of the latter two results in rectangular clusters of recurrence points). \(^3\) Single, isolated recurrence points indicate that a state is rare or only briefly persists. Diagonal lines, running parallel to the LOI, of length \( l \) occur when part of a trajectory runs almost in the same phase neighborhood as a previous portion for \( l \) segments. Finally, vertical lines indicate time intervals in which a state is either stationary or changes very slowly. Applying RPs to the rotational periodic and quasiperiodic dynamics gives expected results, such as required by the Steinhaus conjecture (three gap theorem). \(^5\) Reference 51 provides an overview of these behaviors in the context of the eccentric annular billiard.

To explore the relationship between chaotic dynamics and geometry, we studied the recurrence properties of the long-term motion launched as close to the unstable period-2 orbit as numerically possible. RPs of a \( 10^3 \) iteration trajectory for \( a = 0.2, \epsilon = 1.01, 1.1, 1.2, 1.3 \) are illustrated in Fig. 6.

Although many tools already exist to quantify most features in the RPs presented, two of which are examined in Appendix C, we carry out a simple analysis by introducing a new time measure, \( N_{\epsilon} \): The number of collisions with the outer boundary before the particle exits its initial \( L_\infty, \epsilon \)-neighborhood for the first time. This manifests as the black box in the bottom left-hand corner of all four plots in Fig. 6.

\( \text{FIG. 5. (1)–(4) Successive magnifications of the archipelago seen in the top left of the Poincaré section obtained for } \epsilon = 4, a = 0.2 \text{ (Fig. 4, bottom, center).} \)
Figure 6. Recurrence Plots (RPs), visualizing the dynamics resulting from the unstable period-2 orbit for different elongations. $a = 0.2$. Top left: $e = 1.01$, where the measure of interest $N_\epsilon$ is indicated. Top right: $e = 1.1$. Bottom left: $e = 1.2$. Bottom right: $e = 1.3$ for which dynamical transitions, as indicated by the variation of density of recurrence points, are clearly present.

Figure 7 shows the evolution of $N_\epsilon/N_\circ$, for a fixed value of $N_{\text{col}} = 10^4$, for different ellipse parameters. For all values of $a$ considered, as $e \to 1$, so does $N_\epsilon/N_\circ$. This is because the initial period-2 orbit approaches stability, as demonstrated in Appendix B; therefore, it always remains in its initial neighborhood. Similarly, as $e \to 1/a$, $N_\epsilon/N_\circ$ again approaches unity as the period-2 orbits approach stability. This can be expected as, intuitively, the region becomes more confined, the distance the particle traverses between each collision approaches zero, meaning that small deviations from the initial conditions will have an increasingly negligible effect on the orbit stability matrix (see Appendix B, Sec. A2).

The most important feature of the main plot in Fig. 7 is the transition of $N_\epsilon/N_\circ$ from smooth to rough. This happens at different values of elongation for each value of semi-minor axis, $a$. 
and is insensitive to changes in numerical precision. To understand this phenomenon, recall that the trajectory begins as the unstable period-2 orbit; i.e., the particle collides with the boundary at every other iteration. When \( e = 1 \), \( N_\epsilon = N_e = N_{col}/2 \). As \( e \) increases, \( N_\epsilon/N_e \), smoothly evolves until a critical value of the elongation, \( e = e_{crit}(a) \). This critical geometry causes a non-zero probability that the particle, having just collided with the circular outer boundary, will miss the ellipse and collide again with the outer circle, therefore breaking the parity condition. The first time this event occurs in a trajectory will be referred to as the endogenous escape event or \( \delta \) for brevity. This must not be confused with the escape events normally studied in open billiard systems, as the hole through which the particle is escaping in this case is intrinsic to the system's phase space. The true value of \( e_{crit} \) depends neither on \( e \) nor on the numerical precision chosen. \( e_{crit}(a) \) is defined as the elongation for which the following is true:

\[
\lim_{N_{col} \to \infty} \left( N_e - \frac{N_{col}}{2} \right) = 1.
\]

To estimate \( e_{crit} \) for a given value of \( a \), we used the following algorithm. The initial value of \( e \) is set to \( 1 + \delta e \) with \( \delta e = 0.02 \) in the results presented, and a trajectory of \( 2 \times 10^6 \) collisions is launched near the unstable period 2 orbit. If \( N_e = N_{col}/2 \), the current value of \( e \) is increased by \( \delta e \) and another trajectory is launched. This is continued until \( N_e > N_{col}/2 \) and \( e_{crit} \) is taken as the current value of \( e \).

Due to finite simulation times, any obtained value of \( e_{crit} \) will inevitably be an overestimation. Figure 8 shows that \( e_{crit}(a) \) marks the elongations that permit two classes of behavior by reconstructing, and including as insets, the long-term trajectories of the unstable period-2 orbit for values below and above \( e_{crit}(a) \). When \( e < e_{crit}(a) \), the trajectory explores an extended stochastic region that is bounded by a rotational KAM curve/surface. This rotational curve is destroyed at \( e = e_{crit}(a) \); i.e., holes are created that allow the trajectory to escape its reduced chaotic portion of the phase space after a finite number of iterations. By simple inspection of Fig. 8, no other similar transitions are observed for elongations beyond the critical value.

**FIG. 7.** The ratio of the number of collisions within the initial neighborhood of the unstable period-2 orbit, \( N_\epsilon \), to the total number of collisions with the outer circular boundary, \( N_e \), as a function of the elongation, \( e \). A logarithmic scale is used to show the transition point clearly on each curve. X-axis: The elongation of the inner scatterer for different values of \( a \), each identified by a color in the legend. In each case, total trajectory length, \( N_{col} = 10^4 \).

**FIG. 8.** Evolution of \( e_{crit}(a) \) for \( a \in (0, 1) \). Trajectories of length \( N_{col} = 2 \times 10^4 \) were used to obtain each point. The curve indicates the elongation at which the final rotational curve is destroyed as a function of \( a \). The subfigures show the Poincaré section occupation of the long-time chaotic trajectory (\( N_{col} = 10^4 \)) for elongations just above and below \( e_{crit} \): (a) \( a = 0.1 \) below: \( e = 1.320 \) and above: \( e = 1.321 \), (b) \( a = 0.5 \) below: \( e = 1.178 \) and above: \( e = 1.179 \), (c) \( a = 0.9 \) below: \( e = 1.068 \) and above: \( e = 1.069 \).
For values of elongation below the critical value, $\epsilon_{\text{crit}}(a)$, the trajectories exploring the bounded chaotic region are clearly still subject to both position and momentum diffusion, yet under the constraint that the point particle will always alternately collide with the circle and the ellipse. In this case, one may symbolize the trajectory, $T$, as

$$T = [\ldots, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \ldots]$$

where $c$ and $e$ denote collisions with the circle and ellipse, respectively. For $\epsilon > \epsilon_{\text{crit}}$, this constraint no longer applies. The moment this parity condition is broken implies the dynamical transition has occurred and is characterized by $\epsilon^*$, i.e., the first passage of the particle’s trajectory from one major fractal subset to the rest of the chaotic phase space. In this case, the trajectory may now be symbolized as

$$T = [\ldots, \epsilon_{\text{crit}} - 4, \epsilon_{\text{crit}} - 3, \epsilon_{\text{crit}} - 2, \epsilon_{\text{crit}} - 1, \epsilon_{\text{crit}}, \epsilon, \ldots].$$

The first, bold, consecutive $\epsilon$ represents $\epsilon^*$ by hitting the outer circular boundary twice in a row, which, as before specified, never occurs when $\epsilon < \epsilon_{\text{crit}}(a)$. We note that $\lim_{\epsilon \to \epsilon_{\text{crit}}} n_{\text{hit}} = \infty$. Without explicit numerical calculation, it seems impossible to know the symbolic order following $\epsilon^*$. The set $[\epsilon_1, \epsilon_2, \epsilon_3, \ldots]$ is defined as the first, second, third, etc. times the trajectory consecutively hits the outer boundary after $\epsilon^*$.

V. STUDY OF $\epsilon^*$

To characterize the endogenous escape event, we introduce the angle, $\theta_{\text{esc}}$, at which the escaping segment comes closest to the ellipse, and the distance of the closest approach, $d_{\text{esc}}$, shown in Fig. 9, is presented in Figs. 10–20. These quantities were obtained by noting that, at the point of the closest approach, the trajectory lies parallel to the ellipse at the closest point on the ellipse circumference. Appendix D provides the geometrical derivation of these quantities and further details.

A. Numerical procedure

To ensure that the results presented reflect the global properties of the chaotic region of the phase space, the $N_{\text{sample}} = 10^4$ trajectories studied were launched from slightly different initial conditions, within the neighborhood of the unstable period 2 orbit. The starting position on the outer boundary is defined by

$$\theta_k = (-1)^k \frac{\pi}{2} + \epsilon_0 \eta_k,$$

where $1 \leq k \leq N_{\text{sample}}$. The starting orientation is given by

$$\beta_k = \epsilon_\beta \eta_k,$$

where $\epsilon_0 = 10^{-10}$, $\eta_0, \eta_k$ are random numbers chosen from independent random distributions between $-1$ and $+1$; i.e., the starting position of the two unstable orbits has given very slight, unbiased variations, which will cause the trajectories to exponentially deviate over a finite time and, therefore, follow very different trajectories within the reduced chaotic phase space before escaping.

The following results displayed only minor variations upon setting $\epsilon_0$ or $\epsilon_\beta$ to zero.

B. Results

Heatmaps showing the distributions of $\theta_{\text{esc}}$ and $d_{\text{min}}$ for $\epsilon^*, \epsilon^{10}$, and $\epsilon^{20}$ are presented in Figs. 10, 12, 14, 16, 18, and 20. Results for $a = 0.1, 0.5, 0.9$ are collected for two values of $\epsilon$ above and approaching $\epsilon_{\text{crit}}$. Survival probabilities before $\epsilon^*$ are presented in Figs. 11, 13, 15, 17, 19, and 21. These figures also show a semi-log linear fit, $\log(P_{\text{survival}}) = A n + B$, over the domain demarked by the vertical red dashed lines, using the method of least squares. Both $A$ and $B$ are given within the figures. $n_{\text{min}}$ is the minimum recorded value before any escape and so may be thought of as a deterministic dead time. No power-law behaviors were observed. The heatmaps were constructed by discretizing the $\theta_{\text{esc}}$ and $d_{\text{min}}$ values for each geometry studied and plotting the correspondences between the two values on a 2D histogram. The global heatmaps were made with a resolution of $L \times L$ cells where $L = 500$. A colorbar is used to count the number of events recorded per “cell.” Units are given as “counts per bin” $\text{cph}$. The following results are symmetric around $\theta_{\text{esc}} = 0, \pi$ and $\theta_{\text{esc}} = -\pi/2, \pi/2$.

As it is challenging to fully characterize the forms displayed by the results, we focus on their common properties.

In every case, the distribution of $\theta_{\text{esc}}$, at $\epsilon^*$, is unintuitive and even, as for Fig. 16(a), counter-intuitive. One might expect $\epsilon^*$ to occur when the distance between the inner and outer boundaries is maximum, i.e., at $\theta_{\text{esc}} = 0$; yet here, it is clearly a minimum.
FIG. 10. Heatmaps of $\theta_{\text{esc}}$ against $d_{\text{min}}$. $a = 0.1, e = 1.4$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 11. Survival probability on a semi-log scale. $n_{\text{min}} = 239, a = 0.1, e = 1.4$.

FIG. 12. Heatmaps of $\theta_{\text{esc}}$ against $d_{\text{min}}$. $a = 0.1, e = 1.321$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 13. Survival probability on a semi-log scale. $n_{\text{min}} = 83,759, a = 0.1, e = 1.321$. 
FIG. 14. Heatmaps of $\theta_{esc}$ against $d_{\text{min}}$. $a = 0.5$, $e = 1.2$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 15. Survival probability on a semi-log scale. $n_{\text{min}} = 469$. $a = 0.5$, $e = 1.2$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 16. Heatmaps of $\theta_{esc}$ against $d_{\text{min}}$. $a = 0.5$, $e = 1.178$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 17. Survival probability on a semi-log scale. $n_{\text{min}} = 52137$. $a = 0.5$, $e = 1.178$. 

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FIG. 18. Heatmaps of $\theta_{\text{esc}}$ against $d_{\text{min}}$. $a = 0.9, \epsilon = 1.1$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 19. Survival probability on a semi-log scale. $n_{\text{min}} = 495. a = 0.9, \epsilon = 1.1$.

FIG. 20. Heatmaps of $\theta_{\text{esc}}$ against $d_{\text{min}}$. $a = 0.9, \epsilon = 1.069$ for the 1st, 10th, and 50th escape events (a)–(c).

FIG. 21. Survival probability on a semi-log scale. $n_{\text{min}} = 48891. a = 0.9, \epsilon = 1.069$. 

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VI. DISCUSSION AND CONCLUSION

We introduced a new 2D billiard that exhibits mixed dynamics. As well as presenting complete Poincaré sections, recurrence plots (RPs) were applied to the long-time dynamics resulting from unstable period-2 orbits for different elongations. The study of the RP features led to the identification of a new quantitative measure, which provided strong numerical evidence for the existence of a set of critical elongations, shown in Fig. 8, marking the destruction of the final rotational transport barrier in the phase space, i.e., the two main portions of the chaotic region of the phase space connect, causing a transition to global chaos.

The subfigures within Fig. 8 clearly show that the frequencies of the final destroyed rotational transport barriers correspond to different values of $\beta$. Contrary to the result obtained for the transition to global chaos in the standard map, we postulate that each critical value of elongation is directly associated with a different noble fraction, which defines the frequency of the final destroyed rotational transport barrier. The evidence suggests a connection between the set of noble numbers and the critical ellipse geometries, with deep consequences for the global structure of the chaotic phase space. To further explore this phenomenon, in the context of the system’s symmetries, one would need a Poincaré first return map, which, until now, is unobtainable by known means.

The endogenous escape event, $\mathcal{E}$, that signals the transition to global chaos is illustrated in Fig. 9. It is so named to distinguish it from previously studied escape events that consider the first passage through some pre-defined hole in an open system. For example, survival probabilities of dynamical escape events have been previously studied in the case of “open billiards,” where a MUPO in a mushroom billiard whose stem is replaced by a hole leads to a contribution to the survival probability proportional to $1/t$ as $t \to \infty$. However, in this system, the “holes” created are, unlike in open billiards, intrinsic to the phase space arising from the geometry considered. Furthermore, the diffusive behavior of the reduced chaotic dynamical system’s components means that it would be probably incorrect to try to make an analogy with the previously studied MUPO escape probability distributions. It is, however, surprising that no power-law distributions are at all observed as one would expect the presence of an infinite hierarchy of cantorous regular regions at the boundary between the two connected chaotic phase regions. The results obtained for different ellipse parameters, and, therefore, different times before the escape event, over which the survival probability is computed, mean that one can be sure that the decaying exponential distributions are not masking some longer time power-law distribution. Further results show that the distributions of both the angle of escape and the distance of the closest approach behave counter-intuitively at the moment of escape, and for several iterations after, before relaxing to distributions, one would expect for the chaotic behavior in a mixed dynamical system.

Initial investigations of the 3D version of the Iris billiard, following approaches used in Ref. 61, show that an analogous critical geometric property discussed in this paper persists. It would, therefore, be natural to carry out a similar characterization of the 3D escape event $\mathcal{E}$. It is not clear how to obtain the observable escape features in 3 dimensions by similar arguments as employed here. It would be interesting, and straightforward, to carry out a comparative study on the same phenomenon and resulting observables as studied here in the context of the eccentric annular billiard. If the transition to global chaos is again observed, it would then be possible to study the Poincaré map, which has already been obtained, in
order to identify the periodic orbits that become unstable at the critical perturbation value via Greene’s residue criterion. Finally, it would be of great interest to perform superconducting microwave resonator experiments, such as those previously used to explore chaos-assisted dynamical tunneling, with an Iris domain, or its desymmetrized version. These could be carried out for precisely machined geometries close to the critical values presented here. Such experiments could facilitate a detailed study of transport between the wave analogs of the parts of the phase space that always sometimes or never impinge upon the central ellipse.

ACKNOWLEDGMENTS

We thank Maxime Brunet for performing preliminary simulations of the Iris billiard.

APPENDIX A: DERIVATION OF CONDITIONS PERMITTING RATIONAL \( \omega_1/\omega_2 \) ORBITS

A rational \( \omega_1/\omega_2 \) orbit, where \( \omega_1 < \omega_2 \) defines a \( \omega_2 \) polygon whose radius of intersection of the enclosed caustic formed by its rotation through \( \theta \in [0, 2\pi] \) is as follows:

\[
    r_{\omega_1,\omega_2} = \left| \cos \left( \frac{\omega_1 \pi}{\omega_2} \right) \right|. \tag{A1}
\]

Considering the parametric form of the ellipse and applying simple geometric arguments returns the following condition for intersection:

\[
    b^2 \sin^2 \theta + a^2 \cos^2 \theta - r_{\omega_1,\omega_2}^2 \geq 0, \tag{A2}
\]

which is true for all values of \( \theta \) iff \( r_{\omega_1,\omega_2}^2 \leq a^2 \). Conversely, there are no intersections for any value of \( \theta \) when \( r_{\omega_1,\omega_2}^2 \geq b^2 \). In the regime \( a^2 \leq r_{\omega_1,\omega_2}^2 \leq b^2 \), to determine if there is an intersection, one must solve

\[
    b^2 \sin^2 \theta + a^2 \cos^2 \theta - r_{\omega_1,\omega_2}^2 = 0, \tag{A3}
\]

returning

\[
    \sin^2 \theta = \frac{r_{\omega_1,\omega_2}^2 - a^2}{b^2 - a^2}. \tag{A4}
\]

Therefore, the conditions for intersection with the inner ellipse are, therefore, met by

\[
    |\theta - \pi/2| \leq \arccos \sqrt{\frac{r_{\omega_1,\omega_2}^2 - a^2}{b^2 - a^2}} \tag{A5}
\]

or, by the symmetry of the system,

\[
    |\theta - 3\pi/2| \leq \arccos \sqrt{\frac{r_{\omega_1,\omega_2}^2 - a^2}{b^2 - a^2}}. \tag{A6}
\]

In order for a rational \( \omega_1/\omega_2 \) orbit to exist, one must avoid intersections for all \( \theta + 2\pi n/\omega_2 \forall n \in \{0, \ldots, \omega_2 - 1\} \). Therefore, the following condition must be met:

\[
    \arccos \sqrt{\frac{r_{\omega_1,\omega_2}^2 - a^2}{b^2 - a^2}} < \frac{\pi}{\kappa(\omega_2) \omega_2}, \quad \omega_2 \in \{3, 4, 5, \ldots\}. \tag{A7}
\]

where

\[
    \kappa(\omega_2) = 1 + \omega_2 \mod 2. \tag{A8}
\]

Equation (A7) may be finally rearranged to give

\[
    \frac{b}{a} \geq \varepsilon_{\min} = \left[ \frac{1}{\cos^2(\pi/\kappa(\omega_2) \omega_2)} \left( \frac{r_{\omega_1,\omega_2}^2}{a^2} - 1 \right) + 1 \right]^{1/2}. \tag{A9}
\]

APPENDIX B: STABILITY ANALYSIS OF PERIOD 2 ORBITS

Each orbit within the billiard is an ensemble of alternating straight lines and reflections. To describe the effect of changing the initial conditions on a trajectory of \( n \) segments, we construct a stability matrix, \( M_n \), such that

\[
    \begin{pmatrix} \delta\theta_n \\ \delta\beta_n \end{pmatrix} = M_n \begin{pmatrix} \delta\theta_0 \\ \delta\beta_0 \end{pmatrix}. \tag{B1}
\]

Slight changes, \( (\delta\theta_0, \delta\beta_0) \), made to the initial position perpendicular to the direction of motion and to the initial direction, respectively, at \( (\theta_0, \beta_0) \), through the operation of \( M_n \), produce the resultant deviations \( (\delta\theta_n, \delta\beta_n) \). \( M_n \) is computed as the product of matrices describing the effect of the deviations to each straight line segment and reflection of a trajectory resulting from a slight change in its initial conditions,

\[
    M_n = \prod_{i=0}^{n-1} \begin{pmatrix} l_i & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & \cot \beta_i \end{pmatrix}, \tag{B2}
\]

where \( l_i \) is the length of the \( i \)th segment and \( R_i \) is the radius of curvature of the inner or outer boundary at the \( i \)th collision. Since the system is conservative, the mapping describing the evolution of the trajectory is area preserving, i.e.,

\[
    \text{Det}M_n = 1. \tag{B3}
\]

Eigenvalues of the stability matrix, and, therefore, the properties of the trajectory, depend only upon its trace. To best obtain the analytical properties of the trajectory, its residue, \( R \), as defined by Greene, is used, where

\[
    R = \frac{1}{4} (2 - \text{Tr}M_n). \tag{B4}
\]

The values of \( R \) determined by the value of \( \text{Tr}M_n \) characterize the analytic properties of a trajectory as follows:

\[
    \begin{align*}
        R < 0 & \quad \text{hyperbolic} \\
        R = 0 & \quad \text{marginally stable} \\
        0 < R < 1 & \quad \text{stable} \\
        R > 1 & \quad \text{reflection hyperbolic}
    \end{align*} \tag{B5}
\]

Hyperbolic trajectories correspond to \( M_n \) having real eigenvalues that are positive (or negative in the reflection case). They always
undergo a significant deviation due to the finite limits of numerical precision. Marginally stable trajectories correspond to eigenvalues of \( \pm 1 \). The stable case yields complex eigenvalues with magnitude unity. Two stable trajectories, with slightly different initial conditions, undergo a linear deviation over time.

The trace of the stability matrix of a single period 2 orbit in the direction of the ellipse’s semi-minor axis is

\[
\text{Tr}(M_2) = \frac{4b (e^2 - 1)}{e^3} + \frac{4}{e^2} - 2. \tag{B6}
\]

\(|\text{Tr}(M_2)| \leq 2 \ \forall a, e\). Therefore, the residue of the orbit always indicates stability (0 < \( R \) < 1).

Along the direction of the semi-major axis, the trace otherwise returns,

\[
\text{Tr}(M_1) = -4b (e^2 - 1) + 4e^2 - 2. \tag{B7}
\]

In this case, the orbit is always hyperbolic (\( R < 0 \)) except for where it yields marginal stability (\( R = 0 \)) in the limiting cases, \( a = b \) and \( b = 1 \). (If \( a = b \), i.e., for a circular inner scatterer, all trajectories from every initial condition are either periodic or quasiperiodic, with conserved angular momentum. If \( b = 1 \), the system becomes two separate crescents.) As time tends to infinity, the stable orbit will continue unchanged, while the unstable orbit, knocked off course by numerical imprecision, will explore the accessible chaotic sea.

**APPENDIX C: RECURRENCE QUANTIFICATION ANALYSIS AS INDICATORS OF STICKINESS**

To study the structures presented by RPs, several measures,\(^{13}\) known as Recurrence Quantification Analysis (RQA), are already in use. We will focus specifically on two in the context of the chaotic dynamics of the Iris billiard. The first measure is the recurrence rate (RR), defined as the percentage of black points in an RP,

\[
RR(\epsilon) = \frac{1}{N^2} \sum_{i,j=1}^{N} R_{ij}(\epsilon). \tag{C1}
\]

This may be better understood as the "sparsity" of the \( N \times N \) binary matrix under consideration. In the limit \( N \to \infty \), \( RR \) is the probability that a state recurs to its phase neighborhood, as demarked by \( \epsilon \).

**FIG. 23.** RP of a 2 \times 10^4 long chaotic trajectory, launched from the unstable period-two orbit, for \( a = 0.9, e = 1.1 \). \( RR_{\text{crit}} = 0.1 \). \( DE_{\text{crit}} = 0.3 \). Three separate regions of interest are selected and marked as I, II, and III.
The second measure is based on the distribution of diagonal lines present within the RP,

\[ P(\epsilon, l) = \sum_{i,j=1}^{N} (1 - R_{i-1,j-1})(1 - R_{i+j+i}) \prod_{k=0}^{l-1} R_{i+k,i+k}. \]  

(C2)

Recurrence plots principally show diagonal lines for periodic and quasiperiodic orbits, as one of length \( l \) shows that a segment of a trajectory is close to another segment from a different time for \( l \) iterations. The trajectory’s determinism (DET) is defined as the percentage of black points belonging to a diagonal line of at least \( l_{\text{min}} \).

\[ \text{DET} = \frac{\sum_{i=1}^{N} \text{IP}(\epsilon, l)}{\sum_{i=1}^{N} \text{IP}(\epsilon, l)}. \]  

(C3)

In what follows, \( l_{\text{min}} = 3 \). For all periodic orbits, \( \text{DET} = 1 \). To apply these measures as indicators of stickiness within the billiard dynamics, windows of size \( l_w = 200 \) will be overlayed with the original recurrence plot. RQA will be applied to each window, and the evolution of the above introduced variables will serve as indicators of dynamical transitions.

An example RP applied to a chaotic orbit of length \( 2 \times 10^4 \) is shown. The measures introduced above are applied as a particularly sensitive measurement of intra-chaotic dynamical transitions. Figure 23 shows the RP of a long, chaotic orbit for \( a = 0.9, c = 1.1 \). The trajectory is then analyzed by applying the previously introduced RQA measures to moving windows of length \( w = 200 \). The selected RQA measures (RR and DET) are monitored with respect to time. When the chaotic trajectory encounters a sticky region, the RR significantly changes as its evolution becomes much more regular. Three examples are highlighted in Fig. 23 as Domains I, II, and III. These correspond to regions where both RR and DET surpass \( \text{RR}_{\text{crit}} \) and \( \text{DE}_{\text{crit}} \) defined as 0.1 and 0.3, respectively. These values were chosen for the purpose of demonstrating the presence and detection of stickiness for one specific geometry and will have to be reevaluated for each different geometry considered. Domain III is of particular interest as it corresponds to a high deviation for only the DET measure, but not for RR. Figures 24–26 show the phase occupation of the sticky orbits observed by the RR and DET measures.

**APPENDIX D: DERIVATION OF \( \theta_{\text{esc}} \) AND \( d_{\text{min}} \)**

Let \( \hat{u} \) be the unit vector specifying the trajectory. For a non-intersecting trajectory, there is a point of the closest approach on the ellipse, \( r_e = (x_e, y_e) \), whose tangent is parallel to \( \hat{u} \); see Fig. 9. Thus,

\[ \frac{dy}{dx} = -\frac{xd^2}{ya^2} = \frac{u_y}{u_x}. \]  

(D1)

Substituting for \( x \) in the ellipse equation, Eq. (3), returns

\[ y^2 = \frac{b^4}{(au_x/u_y)^2 + b^2}. \]  

(D2)

Since there are two points on the ellipse whose tangents are parallel to \( \hat{u} \), we obtain two solutions: \((x_{i1}, y_{i1})\) and \((x_{i2}, y_{i2})\).
The idea of a “measurement” of a set at scale, $\epsilon$, defined by $L_\infty$ is fundamental. For each $\epsilon$, one measures the set in a way that detects irregularities of size $\Delta \epsilon$. Ultimately, we want to know how these measurements behave as $\epsilon \to 0$.

Defining the chaos border as the subset, $B$, for $\epsilon > 0$, we also define the smallest number of sets of maximum diameter $\epsilon$ as $N_B(\epsilon)$. The dimension of $B$ reflects the way in which $N_B(\epsilon)$ grows as $\epsilon \to 0$.

If $N_B(\epsilon)$ even approximately behaves as a power law, i.e.,

$$N_B(\epsilon) \simeq c \epsilon^{-D}, \quad \forall c, D \geq 0,$$

where $c$ is some constant and $B$ is then said to have a “box-counting dimension” $D$. This is solved via

$$\log N_B(\epsilon) \simeq \log(c) - D \log(\epsilon),$$

allowing us to obtain $D$, in the limit, as

$$D = \lim_{\epsilon \to 0} \frac{\log N_B(\epsilon)}{\log(1/\epsilon)}.$$
FIG. 28. Heat maps relating the escape angle $\theta_{esc}$ to the number of trajectory iterations at (a) the first escape event (i.e., the 1st miss), (b) the 10th miss, and (c) the 50th miss. The sample was collected from a set of trajectories launched from the unstable 2 period orbit with slight deviations as given in Eqs. (15) and (16). Geometry of the system is $a = 0.1, e = 1.4$.

FIG. 29. Same as for Fig. 28, but $a = 0.1, e = 1.321$. 
To determine the fractal dimension, grid cells between $L = 500$ and $L = 1500$ were used. The length of each chaotic orbit was $2 \times 10^9$, to ensure a large number of counts per cell, even for larger values of $L$. Using short trajectories to fill the chaotic set results in an anomalous deviation of the border cell count with number of cells $L$ from the expected power law. This is because the sparsity of points recorded within the chaotic phase portion leads to the misidentification of border cells where, in fact, there are none.
FIG. 32. Same as for Fig. 28, but \( a = 0.9, e = 1.1 \).

FIG. 33. Same as for Fig. 28, but \( a = 0.9, e = 1.069 \).

**TABLE I.** Fractal dimensions of the border of the chaotic component for chosen elongations under the critical set.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( e )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.317</td>
<td>( 1.70 \pm 1 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.5</td>
<td>1.175</td>
<td>( 1.62 \pm 2 \times 10^{-3} )</td>
</tr>
<tr>
<td>0.9</td>
<td>1.067</td>
<td>( 1.80 \pm 7 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

**TABLE II.** Evolution of the fractal dimensions for the border of the chaotic component for chosen elongations above the critical set.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( e )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.321</td>
<td>( 1.63 \pm 1 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.5</td>
<td>1.178</td>
<td>( 1.53 \pm 3 \times 10^{-3} )</td>
</tr>
<tr>
<td>0.9</td>
<td>1.069</td>
<td>( 1.95 \pm 1 \times 10^{-2} )</td>
</tr>
</tbody>
</table>
for $a = 0.9$ to be very high, indeed higher than those measured for other ellipse parameters, and close to the upper dimensional limit, 2.

**DATA AVAILABILITY**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**REFERENCES**

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