Nonlinear transient and chaotic interactions in disc brake squeal

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Abstract

In automotive disc-brake squeal, most numerical studies have been focused on the prediction of unstable vibration modes in the frequency domain using the complex eigenvalue analysis. However, the magnitude of the positive real part of a complex eigenvalue is an unreliable indicator of squeal occurrence. Although nonlinearities have been shown to play a significant role in brake squeal, transient nonlinear time domain analyses have rarely been applied owing to high computational costs. Here, the complex eigenvalue analysis, the direct steady-state analysis, and the transient nonlinear time domain analysis are applied to an isotropic pad-on-disc finite element model representing a simple model of a brake system. While in this investigation, in-plane pad-mode instabilities are not detected by the complex eigenvalue analysis, the dissipated energy obtained by the direct steady-state analysis of the model subjected to harmonic contact pressure excitation is negative at frequencies of pad modes, indicating a potential for instabilities. Transient nonlinear time domain analyses of the pad and disc dynamics reveal that in-plane pad vibrations excite a dominant out-of-plane disc mode. For intermittently chaotic pad motion, the disc dynamics is quasi-periodic; and for chaotic motion of the pad, a toroidal attractor is found for the disc’s out-of-plane motion. Nonlinear interactions between the pad and the disc highlight that different parts in a brake system display different dynamic behavior and need to be analyzed separately. The type II intermittency route to chaos could be the cause for the experimentally observed instantaneous mode squeal.

1. Introduction

For the automotive industry, brake squeal noise has increasingly become a source of customer dissatisfaction [1,2]. In North America, up to one billion dollars p.a. was spent on Noise, Vibration and Harshness (NVH) issues while friction material suppliers allocated more than half their budgets to dealing with NVH problems [3]. About 60% of warranty claims concerning the brake corner are due to brake squeal [4]. Despite a good deal of progress being made in the past two decades, the underlying mechanisms of brake squeal are not yet fully understood [2] and prediction of brake squeal propensity remains difficult [5].

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Brake squeal is defined as a self-excited and self-sustained unwanted noise in the audible frequency range above 1 kHz [2]. Low-frequency squeal is usually found below 5 kHz and below the disc’s first in-plane mode [6]. It interacts with more brake components than high frequency squeal which predominantly involves the pads and the disc [1,6,7]. Brake squeal is fugitive and often not repeatable even under apparently identical operating conditions [2]. Reviews of brake squeal and friction-induced noise were given by Kinkaid et al. [1], Akay [1] and Akay [8] while numerical methods for analysing brake squeal were discussed by Ouyang et al. [9], Sinou [10] and Oberst et al. [11]. Basic squeal mechanisms identified include sprag-slip, stick-slip, the negative gradient of the friction coefficient with the sliding velocity and flutter instability owing to mode coupling [1]. The current consensus is that most squeal occurs during steady sliding between the disc and the pad with mode coupling being the dominant mechanism and the brake disc being the dominant sound radiator [1,6,12].

The current industrial practice for brake squeal analysis involves the application of the linear complex eigenvalue analysis (CEA) to numerical finite element models to predict unstable vibration modes. Chern et al. [16] pointed out that the CEA as a linear tool disregards rotor rotation, instant contact stiffness and other trigger mechanisms. The number of unstable vibration modes may be underestimated [10,13] or overestimated [14,15]. Although the linear CEA is not reliable in predicting the occurrence and the magnitude of squeal, it is generally used to complement extensive noise dynamometer tests and to examine options for countermeasures [2].

Numerical transient nonlinear time domain analyses (TDA) are rarely applied owing to high computational costs [9]. By conducting a combined thermal, contact pressure and nonlinear transient analysis using ABAQUS Explicit, AbuBaker and Ouyang [13] showed that a thermal analysis would improve modelling accuracy; however a nonlinear dynamics analysis has not been performed. Nonlinearity and transient processes are essential for the generation of brake squeal [6,17] and additional unstable modes can appear owing to transient vibrations, nonlinear friction or contact interactions (e.g. loss of contact) [10].

Recently, the noise performance of a brake system has been found to be correlated with the degree of nonlinearity [18] and recorded squeal noise has been shown to exhibit chaotic behaviour [19] even though limit cycles and quasi-periodic regimes were dominant. Large amplitude vibrations were generated and the phase space dynamics evolved from a limit cycle over a torus regime to a low dimensional chaotic attractor through the RUELLE–TAKENS route to chaos [19]. Sudden transient periodic bursts of large amplitude radial vibrations were observed in a 4-dof analytical model with sign function (switching) nonlinearity [20]; bursting large amplitude vibrations were also observed in nonlinear transient finite element TDA [21] but not analysed in detail.

The objective of this study is, therefore, to investigate the transient vibration behaviour of an isotropic pad-on-disc model as a simple model of a brake system using a nonlinear TDA. In particular, the interactions between the pad and the disc are examined with regard to their role in generating instabilities. Conventional CEA is compared to the results of the TDA. In addition, the dissipated energy spectra obtained by a analysis (steady-state analysis) of the pad-on-disc model subjected to harmonic contact pressure excitation are contact pressure excitation are used to complement both the CEA and TDA. Results are interpreted as the experimentally observed mechanism of instantaneous mode squeal which is triggered by an impulsive excitation caused by sudden friction release – attachment processes [6].

2. The numerical model

A simple model of a brake system, in the form of a steel pad in sliding contact with an annular disc (pad-on-disc), is depicted in Fig. 1(a); the magnified contact area in Fig. 1(b) with the highlighted top layer of pad elements shows where the uniformly distributed pressure is applied. At each of the four upper corner nodes, the pad is constrained in the tangential (t) and radial (r) direction with no rotational degrees of freedom. The disc is constrained at the inner edge of the disc adjacent to the face opposite to the friction interface and only free to rotate about its z-axis. The measurement points 1, 2 and 3 (Fig. 1(b)) are chosen to investigate the dynamic interplay between different domains and correspond to a node on the disc, a

![Fig. 1. (a) Numerical model of annular disc with simplified brake pad and (b) the magnified area and measurement points 1, 2 and 3 on pad-on-disc system; dimensions in mm; top of pad where uniformly pressure is applied is highlighted.](image-url)
node at the centre of the pad’s top surface and a node in the pad–disc contact interface, respectively. The choice of a
measurement point within each domain does not influence the interpretation of the general dynamic behaviour [22]. This
has been confirmed for several other measurement points but is not presented here for brevity reasons. \( U_i, U'_i \) and \( U''_i \) with \( i \in \{1, 2, 3\} \) being the displacements in the radial (\( r \)-), tangential (\( t \)-) and out-of-plane (\( z \)-) directions respectively with ROT indicating the disc’s rotation about its central axis.

The numerical brake model was simulated using ABAQUS 6.8-4/Standard in the frequency domain [11] and then the
Explicit solver was used to perform a TDA leaving all parameters at their default values. The finite sliding formulation for
frictional surface-to-surface contact calculates the normal and tangential friction forces over the stresses in the contact
interface and permits (partial) separation, sliding of finite amplitude and arbitrary rotation of the contact surfaces and is
expressed in non-symmetric form as a non-regular nonlinearity (partial lift-off) [11, 23–25]. Similar to Massi et al. [25] the
explicit dynamic finite element code uses the principle of virtual work to determine the equations of motion at each
time step.

Young’s modulus, Poisson’s ratio and densities of the disc and the pad are 110 and 200 GPa, 0.28 and 0.30 and 7800 and
7200 kg/m³, respectively. About 23,331 linear hexahedral reduced integrated elements are used because (1) calculations are fast (reduced integration); and (2) as some critical pad modes [26] occur at lower frequencies, larger time-stepping may be used. A mesh independence study in the frequency domain was performed [11] but without considering a partitioned mesh in the contact area as in the subsequent TDA, the mesh would move relative to the fixed global coordinate system. Therefore the whole disc instead of only a partition had to be sufficiently fine meshed.

For the TDA, a smooth transition from the transient to the steady-state self-excitation is approximated by initially
ramping up the velocity from 0 to 10 rad/s in the time interval 0–0.1 s and ramping up the contact pressure from 0 to 1 kPa
in the time interval 0.15–0.25 s (Fig. 2). The time durations for ramping up the velocity and the pressure were chosen to be
non-overlapping and as small as possible [27], but long enough to allow the solver to run stable. The ramp time (swell time)
of the pressure is within the common brake swell time of passenger cars which is between 0.1 and 0.2 s [28].

The TDA simulations were conducted using an arbitrary LAGRANGIAN–EULERIAN formulation in ABAQUS to enhance stability (reduction of numerical chatter) and to avoid critical element distortion. The disc was assigned to the LAGRANGIAN domain while the pad was assigned to the stationary EULERIAN domain. During the first 0.1 s of the rotation, the pad was re-meshed 15 times and afterwards, 100 times per time increment. This substantially increased stability and allowed calculation of long time traces. To further stabilise (e.g. by reducing high frequency numerical chatter) and to reduce initial transients during the first 0.25 s, the linear and quadratic numerical damping parameters \( \alpha_1 \) and \( \alpha_2 \) were set to \( \alpha_1 = 0.24 \) and \( \alpha_2 = 1.6 \) (bulk viscosity). Thereafter these parameters were reduced to their default values of \( \alpha_1 = 0.12 \) and \( \alpha_2 = 1.4 \). If the damping value of the linear parameter was increased to \( \alpha_1 = 0.28 \), the time series was delayed but the dynamics was not changed (i.e., same envelope, same dynamic invariants and same mutual information). If \( \alpha_1 \) was further increased to 0.30 the calculated envelope differed from the undamped case qualitatively. For the quadratic bulk viscosity the critical parameter value was \( \alpha_2 = 2 \) for which significant delays in the time history were observed. Following the linear increase of the velocity and the contact pressure, convergence was only considered to be achieved if the total energy was bounded and the artificial strain energy and damping dissipation (bulk viscosity damping, material damping) was within 1% of the system’s kinetic energy or the strain energy. The effects of the numerical damping for this study are insignificant as the transients (about 0.3 s) are much longer than 0.05 s, 0.2 s and 0.06 s respectively for AbuBaker and Ouyang [24], Massi et al. [25] and Nouby et al. [29]. The simulation ran for about 1.8 s with an average time-step size of 0.002 s.

Fig. 2. TDA time-dependent velocity \( (t_0 = 0 \text{ s to } t_1 = 0.10 \text{ s}) \) and pressure \( (t_2 = 0.15 \text{ s to } t_3 = 0.25 \text{ s}) \).

3. Frequency-domain analysis

A frequency domain analysis was conducted using a CEA and a direct, steady-state analysis of the model, with the brake
pad’s top surface being subjected to harmonic contact pressure excitation of 1 kPa acting in the z-direction. From the direct,
steady-state analysis based on the dynamic virtual work principle, a linearised perturbed solution around the current base
state is obtained to form the response spectra so that the global dissipated energy spectrum can be extracted [11, 26]. Stress (contact) stiffening as well as geometrical nonlinear effects have been included in the nonlinear base-state through a

\[ \frac{\partial U}{\partial t} = \gamma(U, U') \]
nonlinear static analysis. The load and the response define the change from the base state of the system with the convergence of the simulation being controlled by ABAQUS which terminated all analysis steps without errors.

### 3.1. Complex eigenvalue analysis

Pad modes are denoted by \( P_t, P_r \) and \( P_{rot} \) (tangential, radial and rotating in-plane to the disc) and have been found to vary largely with the friction coefficient or contact pressure [26]. Here the friction coefficient was varied from \( \mu = 0.05 \) to \( \mu = 0.68 \). A vibration mode is considered to be unstable if the effective damping ratio \( \zeta \) is negative with \( \zeta = -2 \text{Re}(x)/\text{Im}(x) < \gamma = -10^{-5} \), \( \gamma \) being an arbitrarily small threshold value; \( \text{Re}(x) \) (decay rate) and \( \text{Im}(x) \) (eigenfrequency) denote the eigenvalue’s real and imaginary parts. All pad modes seem to be stable (largely negative real parts) and two unstable disc modes have been identified: the \((m,n,l,q) = (0,4,0,0)\) mode-pair at about 4062 Hz and the \((0,5,0,0)\) mode pair at about 5896 Hz with critical friction coefficients 0.28 and 0.44 respectively (see Fig. 3). In the notation of the modes, \( m, n \) represent the number of out-of-plane nodal circles and diameters and \( l, q \) the number of in-plane nodal circles and nodal diameters of the disc with \( \pm \) indicating a positive and negative travelling wave (a standing wave in a frequency domain analysis with an anti-node (+) or a node (−) being coincident with the contact point of the disc’s split modes) [11,25].

### 3.2. Direct, steady-state analysis (DSA)

Fig. 4(a) depicts the undamped pad’s displacement response at the same pressure of \( p = 1 \) kPa as that for the TDA at measurement point 2 (Fig. 1) in the radial, tangential and out-of-plane directions respectively (\( \mu = 0.05 \)). Fig. 4(a) indicates that at resonances of pad modes, while the in-plane radial \( (P_r) \) and tangential modes \( (P_t) \) dominate the in-plane motion, still a small out-of-plane component exists. In Fig. 4(b) these out-of-plane responses \( (U_2) \) of \( P_r \) and \( P_t \) increase with the friction coefficient. As the pad is coupled via friction to the disc, in real life applications in some circumstances a feedback loop could be established, which would be able to feed energy back into the brake system [30].

Fig. 3. Results of the CEA show (a) the real parts and (b) and (c) the imaginary parts of the \( n=4 \) and the \( n=5 \) modes which destabilise at \( \mu = 0.28 \) and \( \mu = 0.44 \), respectively.
Negative dissipated energy in Fig. 5 for $\mu = 0.05, 0.30$ and $0.46$ and $p = 1$ kPa, displayed as absolute values by diamond markers, indicates that energy is provided by friction to the system\[30\]. The two predicted unstable disc modes ($n = 4$ and $n = 5$) both indicate a provision of energy for $\mu = 0.30$ and $\mu = 0.46$. At 2.568 kHz, energy is also provided to the system for the $n = 3$ disc mode. While the energy provided by the $n = 4$ disc mode decreases at $\mu = 0.46$, it increases for the $n = 5$ disc mode. At 1.14 kHz, the $n = 1$ disc mode is dominant with a pad moving tangentially, providing energy for $\mu \geq 0.3$. However, the radial and tangential pad modes $P_r$ and $P_t$ at around 2 kHz and the rotational pad mode $P_{rot}$ ($ \approx 3.2$ kHz, rotating in-plane about the pad’s $z$-axis), predicted by the CEA as stable, show large frequency intervals of negative dissipated energy (determined by DSA) very similar to the behaviour of the pad-on-plate system investigated by Oberst and Lai\[26\]. These unstable modes might be related to the instantaneous mode squeal experimentally observed by Chen\[6\].

4. Time domain analysis (TDA)

Based on the CEA results (Fig. 3), the following friction coefficients were chosen for the TDA: $\mu = 0.05$ with no instability; $\mu = 0.30$ and $\mu = 0.42$ with still one instability; $\mu = 0.46$ and $\mu = 0.60$ with two unstable disc modes. Firstly the disc dynamics, then the pad dynamics and finally the interactions between the pad and the disc are studied in detail. For the disc, only the acoustically most relevant out-of-plane direction\[1\] is investigated.

4.1. Disc dynamics

Fig. 6 depicts (a)–(d) the time series and (e)–(h) the spectral content (short time Fourier transform, a window length of 256 samples, a large overlap of 254 samples to visualise faster time-varying signals), 1024 FFT lines, a sampling rate of 14 kHz) of the disc’s out-of-plane displacement ($U_z$, Fig. 1). By increasing the friction coefficient from $\mu = 0.05$ (Fig. 6(a) and
(e)) to $\mu = 0.3$ (Fig. 6(b) and (f)), the vibration becomes quasi-periodic. Fig. 6(b) and (f) shows the development of the $n=4$ disc mode at 4062 Hz and the $n=5$ disc mode at 5896 Hz. As the friction coefficient is increased to $\mu = 0.46$ the vibration amplitude grows (Fig. 6(c)). Fig. 6(g) identifies a dominant resonance at 2.28 kHz which is close to that of the pad modes $P_t$ (2.22 kHz) and $P_r$ (2.09 kHz). Also the resonance of the $n=3$ out-of-plane disc mode at about 2.57 kHz (cf. Fig. 5) is highlighted.

Increasing the friction coefficient further to $\mu = 0.60$ does not show major differences within the first 0.8 s with the disc’s out-of-plane displacements of less than 0.1 mm being of the same order of magnitude as reported in the literature [6]. From the spectrograms in Fig. 6, it appears that the changes in frequency over time result from two different effects which highlight the significance of nonlinear time domain analysis. The first frequency change is presumably due to instantaneous local contact stiffness alterations [16] resulting in a frequency shift of about $\pm 137$ Hz around a main frequency of 4115 Hz. It is known that the contact area and pressure distribution play important roles in brake squeal [6]. Here, the partial loss of contact is caused by the coupling between the pad and the disc, resulting in a reduction of contact area and local stiffness changes. A second and larger frequency change happens, starting at about 0.4 s where first the pad mode $P_t$ is dominant and is then accompanied by the vibration of the disc mode $n=1$ (Fig. 6(c)–(d) and (g)–(h)). The development of high amplitude vibrations at this low frequency is caused by the asymmetric time dependent pressure distribution and by applying only one pad. Frequency changes and additional unstable modes (bifurcations) have been reported in experiments and nonlinear transient simulations and are related to nonlinear contact conditions, partial lift-offs and time-dependent behaviour [10,16,19]. The results here indicate that the CEA is consistent with the TDA for the first instability detected at $\mu = 0.3$ but that the second unstable $n=5$ disc mode is according to the TDA not unstable. The TDA shows that the $n=3$ disc mode (resonance with feed-in energy, Fig. 5) which is close to the pad modes (large feed-in energy Fig. 5) first becomes unstable and then after about 0.8 s the $n=1$ disc mode become unstable.

Next a longer time series of the disc $\mu = 0.60$ is analysed using nonlinear time series analysis [19]. The vibration velocity has been shown to be better suited to (a) calculate the recurrence plot quantification analysis measures later on and (b) distinguish the two dynamic regimes $D_0$ and $D_1$ since the mutual information gave higher and different values of the time...
delay $\tau = 0.857 \text{ ms (D}_0\text{)}$ and $0.286 \text{ ms (D}_1\text{)}$ than the displacement’s estimation which had $\tau = 0.143 \text{ ms for both D}_0\text{ and D}_1\text{.}^1$

While theoretically the same attractor should be obtained (irrespective of which phase space variable is used in the embedding process [22]), it is known that in practice e.g. in the presence of noise or simply higher order dynamics some variables are more suited for the embedding process [32]. With the knowledge of which embedding parameters to take the attractors of the displacement signal could also be reconstructed.

Fig. 7 gives the out-of-plane velocity of the disc ($\mu = 0.60$), its maximal amplitude of 0.254 mm/s being also of the same order of magnitude reported in the literature [31]. Two dynamic regimes $D_0$ and $D_1$ are identified. $D_0$ is found in the time traces for both $\mu = 0.46$ and $\mu = 0.60$ with a maximum velocity amplitude of less than 0.2 mm/s, which is exceeded in $D_1$ (Fig. 7).

Next, instead of plotting a spectrogram, the power spectral density (PSD) in regimes $D_0$ and $D_1$ ($\mu = 0.60$) is analysed in Fig. 8 as the heights of some resonance peaks especially for smaller differences (periodogram, 2048 samples, 50% overlap) are better distinguished. The dynamic regime of the disc $D_1$ is on average about 19.5 dB higher than that of $D_0$ with a maximum difference of about 40 dB. The dominant frequency for $D_0$ is around 2.25 kHz, and lies between the pad modes $P_1$ and $P_1$ ($\approx 2.1 \text{ kHz}$) and the $n=3$ disc mode ($\approx 2.57 \text{ kHz}$). The dominant frequency for $D_1$ is around 950 Hz, hence close to the $n=1$ out-of-plane disc mode (Fig. 4). Phase-space plots of the vibration displacement of the disc in the out-of-plane direction ($U^1_P$) for the regime $D_0$ and rendered attractors for the regime $D_1$ are presented in Fig. 9. Shortly after the full contact pressure of 1 kPa is established originating from minuscule periodic self-excited vibrations [14] for a very short time ($0.26–0.2615 \text{ s}$), a limit cycle (Fig. 9(a)) is formed. A vortex regime owing to a secondary Hopf-bifurcation follows (Fig. 9(b)). The trajectory spirals out from the limit cycle only to form, in the end, a quasi-periodic 2-torus$^2$ (in $D_0$, Fig. 9(c)). The maximum Lyapunov exponent is zero [32] and the torus looks like a flat band structure similar to that identified in [5] but without folding operations having set in yet. As the disc dynamics proceeds from a limit cycle (Fig. 9(a)) to a toroidal attractor (Fig. 9(d)), the phase-space diameter ($D_0$) increases (cf. [25]). However, from the phase plot, it is not possible to determine if the evolved torus is actually unstable and will lead to squeal. Therefore next a recurrence plot analysis for the disc is conducted [18,33] to confirm whether the system destabilises.

Fig. 10(a) shows the distance plots of the time series of disc’s vibration for $\mu = 0.60$ in the time intervals $0–0.85 \text{ s}$ (including $D_0$) and $1.8 \text{ s (D}_1\text{)}$. Before the time trace cycles onto a toroidal manifold ($D_0$), a limit cycle with a very small amplitude is established. Then in $D_0$, a dynamically stable regime is formed indicated by straight diagonal lines in the distance plot in Fig. 10. Straight non-disrupted diagonal lines indicate either periodic or quasi-periodic dynamics [34]. In Fig. 10(b), $D_1$ can be further subdivided into subsets $D_{1a}$ to $D_{1c}$ which is useful to study different (unstable) periodic orbits [32] of the attracting set. Unstable periodic orbits act similar to a skeleton of a chaotic attractor [35]. The lines in the sub matrices of the distance plot in Fig. 10(b) are sometimes broken, indicating recurrent but chaotic states [34]. The unstable periodic orbits (recurring not repeating trajectories, see [35]) are depicted in Fig. 11 and highlight the two different regimes $D_{1a}$ (equal to $D_{1c}$) and $D_{1b}$ discernible in the distance plots (Fig. 10): in regime $D_{1b}$ the torus changes to a slightly higher second frequency, maintaining its fundamental frequency (distance between diagonal lines in Fig. 10(b) gets found within the trajectories of the other two regimes). The Poincare section (cut at intersection A of the attractor) indicates a breakup of the torus (cf. [36]) and the onset of chaotic vibrations.

Recurrence plot quantification analysis measures, used recently in analysing recorded microphone data in brake squeal testings [18,19] and friction data [33], were applied to the out-of-plane velocity signal of the disc (Fig. 12). A window of 500 samples was moved with a unity shift, an embedding dimension of $m=6$, a delay of $\tau = 0.31$ ms and an epsilon 7.2% of the maximum phase space diameters standard deviation [34]. The two dynamic regimes $D_0$ (band structure in Fig. 9(c)) and $D_1$ (torus in Fig. 9(d)) are marked in Fig. 12. Fig. 12(a) shows that the ratio of determinism and recurrence rate [18] DET/RR increases largely with time indicating higher irregularity and a change from a periodic/quasi-periodic regime to a chaotic regime. The same holds true for the divergence (DIV) which is the reciprocal value of the maximum diagonal line length in a recurrence plot. The DIV in the regime $D_0$ is less than 0.1 and close to zero most of the time but increases to approx 0.5 in $D_1$, a strong indication of instability (cf. [18]). The determinism (DET) and laminarity (LAM) in Fig. 12(c) both drop with approaching instability.

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1 The delay $\tau$ is needed in the phase space reconstruction process of a scalar time series using delay embedding [22].

2 Two frequencies involved, see spectrogram in Fig. 6(g) @ $\mu = 0.46$. 
In summary, the high values of DIV, DET/RR and the low values of LAM and DET indicate that the behaviour of the torus in Fig. 9(d) is unstable. The correlation dimension calculated over the correlation integral of the regimes $D_{1a} - D_{1c}$ is about $D_2 = 2.20 \pm 0.12 \in CI_{90} = [1.4, 3.1]$ (n = 100) and the maximum Lyapunov exponent, estimated by the algorithm of Sano and Sawado and the algorithm of Kantz [32] is 0.12 and 0.10 $\pm$ 0.03 respectively. The positive maximum Lyapunov exponent, the
non-integer correlation dimension and the torus attractor suggest an instability due to toroidal chaos [35,36]. The chaotic dynamics together with a growing phase space diameter suggests that squeal develops.

4.2. Pad dynamics

Fig. 10. (a) Distance plot in which two zones emphasised (boxed areas) and analysed in interval with vortex and torus regime ($D_0$) (delay of $\tau = 1.54E-4$ s, embedding dimension $m = 8$) and (b) distance plot of regime $D_1$ which is subdivided into $D_{1a}, D_{1b}$ and $D_{1c}$; it appears that $D_{1a} = D_{1c}$.

![Distance plots](image)

Fig. 11. Torus with unstable periodic orbits and Poincare section (A).

non-integer correlation dimension and the torus attractor suggest an instability due to toroidal chaos [35,36]. The chaotic dynamics together with a growing phase space diameter suggests that squeal develops.

4.2. Pad dynamics

Fig. 13(a)-(f) shows the time traces (1.75 s long) and (g)-(l) the spectrogram of pad displacements ($U_z^2, U_z^2$, Fig. 1) for $\mu = 0.46$ and $\mu = 0.60$. Two regimes are studied in more details: $P_0 \in [0.25, 0.6]$ s and $P_1 \in [1.1, 1.75]$ s corresponding to disc regimes $D_0$ and $D_1$. For $\mu = 0.46$ the time traces of Fig. 13(a)-(c) show intermittent bursts especially in the z-direction and the spectrogram in Fig. 13(g)-(i) highlights a dominant frequency component at around 2 kHz with a component at 1 kHz becoming equally strong after 1.6 s (Fig. 13(l)). For $\mu = 0.60$ after a transient period, the same intermittent dynamics as for $\mu = 0.46$ found in $P_0$ is followed by large-amplitude vibrations in $P_1$ (Fig. 13(f), $U_z^2$). For $\mu = 0.60$ the dynamics behave similarly with a resonance at around 1 kHz appearing already after 0.8 s (Fig. 13(l)). Based on the frequency domain analysis, the dynamics of the disc and the inspection of the operating deflection shapes, the resonance at around 1 kHz can be assigned to the $n=1$ disc mode, $n=1$ disc mode.

Fig. 14(a) displays the PSD estimates for regimes $P_0$ and $P_1$ with the friction coefficient $\mu = 0.60$. For the regime $P_1$, the spectrum is much higher than for $P_0$ with only one clear frequency peak at 992 Hz as opposed to several distinct frequency
peaks for the disc regimes $D_0$ and $D_1$ (Fig. 8). In regime $P_1$ with a friction coefficient of $\mu = 0.46$ the phase-space plot and Poincare section (marked by cut A) of the pad’s out-of-plane displacement plotted in Fig. 14(b) using delay embedding [32].

The two-dimensional Poincare section is fractal and shows a very similar attractor to the original three-dimensional attractor, thus indicating self-similarity [37]. Further reducing the Poincare section into Lorenz cross sections [32] is avoided for the following reasons: (1) the computational time needed increases significantly; and (2) for each projection, a loss of accuracy of approximately one order of magnitude will degrade the quality of the fractal map generated [32]. For the regime $P_0$, no clear structure is identified in the phase-space plot (not shown here) probably owing to the intermittent behaviour of the displacement ($U_2$).

The Kaplan–Yorke dimension $D_{KY}$ as an estimate of the fractal dimension (complexity) and the Lyapunov exponents $\lambda_i$ as estimates of the divergence rate (predictability) were estimated by means of the algorithm of Sano and Sawada [38] to be

$$D_{KY} = 2.483 \quad \lambda_1 \in (0.13, 0.00, -0.20) \quad (\sum \lambda_i = -0.07)$$

indicating that the dynamics can be described by using a model with at least three independent variables. The attractor’s dynamics is chaotic [32] with a maximal Lyapunov exponent of $\lambda_{max} = \lambda_1 = 0.13 > 0$. For the second largest Lyapunov exponent $\lambda_2 \approx 0$, the attractor is stable and the phase-space trajectories contract for the negative third Lyapunov exponent $\lambda_3 = -0.20$ [32].

4.3. Interactions between pad and disc

Since the dynamics of the pad and the disc are coupled over the friction contact interface, the interactions between the pad and disc are examined by analysing the dynamics at measurement point 3 (Fig. 1(a)) depicted in Fig. 15 for the radial, tangential and out-of-plane displacements. By increasing $\mu$, the displacements increase. For $\mu = 0.30$, the out-of-plane displacement ($U_3$) becomes dominant. For $\mu = 0.42$, peaks of $U_3$ appear almost regularly after an average of $12.4 \pm 2.9$ ms ($\approx 80 \pm 25$ Hz).

For $\mu = 0.46$, peaks of $U_3$ become less frequent and intermittent and are up to four times larger than for $\mu = 0.42$. This intermittent behaviour is referred to as the regime $PD_0$ (pad–disc interaction). Two intervals, $I_1$ and $I_2$, are also marked with dashed boxes in Fig. 15(I) in order to analyse in detail the characteristics of the low amplitude but regular and high amplitude intermittent dynamics. The in-plane motions ($U_1$ and $U_2$) are modulated with a low frequency component (the pad moves radially $2.614 \mu$m towards the outer rim of the disc and follows it $3.324 \mu$m in the sliding direction). This low frequency modulation is not observed if the friction coefficient is smaller than observed if the friction coefficient is smaller than $\mu = 0.46$, that is if only one mode is predicted to be unstable (Fig. 3). The low frequency modulation of the in-plane and out-of-plane pad dynamics indicates that some transitional processes are acting.

Although $U_2$ obtained by the direct, steady-state analysis increases with the friction coefficient (Fig. 4(b)), this dependency becomes clearer in Fig. 15(I) for $U_2$. However, by harmonically exciting the pad (DSA) it is possible to show

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It took already around 324 h to calculate a time series of 1.8 s using eight processors in a HP workstation Z600, 24 GB RAM, Intel Xeon CPU E5520 @ 2.27 GHz.
some of the features of self-excitation (TDA) which lead to dig-in and release processes and instantaneous contact stiffness changes similar to Chen’s experimental observations [6].

Fig. 16(a)–(f) displays the spectrograms of the radial ($U^2_r$), tangential out-of-plane ($U^2_\theta$) displacements depicted in Fig. 15 for friction coefficients $\mu = 0.05, 0.30, 0.42$ and 0.46. For $\mu = 0.05$, only low-frequency displacements owing to disc rotation are visible (Fig. 16(a)–(c)) but at $\mu = 0.30$ (Fig. 16(d)–(f)), the frequency of the unstable $n=4$ disc mode becomes dominant. The frequency changes around 4115 ± 120 Hz with a maximum shift of up to 137 Hz could be attributed to instantaneous contact stiffness changes as a result of asymmetric contact pressure distributions, local sticking, sliding or partial lift-offs.

Also visible is a stronger frequency component of the unstable $n=5$ disc mode at 5980 Hz (Fig. 16(f)). This component is stronger in $U^2_\theta$ but is almost non-existent in $U^2_r$ (Fig. 16(e)). For $\mu = 0.42$ (Fig. 16(g)–(i)), $U^2_r$ and $U^2_\theta$ show intermittent
high-frequency content, with a still dominant unstable $n=4$ disc mode. The high-frequency components may be due to friction contact which is known to induce higher harmonics [8,19]. For the out-of-plane displacement ($U_3^z$), mostly the frequencies below 4 kHz are excited. For $\mu=0.46$, while unstable $n=4$ and $n=5$ disc modes are visible, $U_3^z$ is dominated by the pad modes $P_r$ and $P_t$, at around 1.04 kHz corresponding to the stable $n=1$ disc mode.

As the out-of-plane pad vibrations may excite directly the out-of-plane disc vibrations [6], Fig. 17(a) displays the PSD of $U_3^z$ using WELCH’S averaged modified periodogram method of spectral estimation. It is obvious from Figs. 17(a) and 8 that the spectrum lifts up more for the contact interface (point 3) than for the disc (point 1). The frequency vector is segmented into eight sections of equal length with 50% overlap, discarded trailing entries then each segment is windowed by means of a HAMMING window, discarding the first 0.35 s as transients. WELCH’S method of spectral estimation is applied to reduce the noise in order to obtain a better estimate of the power levels and the fundamental frequency.

For $\mu=0.05$, a dominant vibration occurs at frequencies below 1 kHz accompanied by a weak pad-mode vibration at around 2.22 kHz. For $\mu=0.30$, a resonance close to the unstable $n=4$ disc mode at 4 kHz dominates. For $\mu=0.42$, the
unstable $n=5$ disc mode at 5.98 kHz and the pad modes ($P_t$ at 2.08 and $P_r$ at 2.1 kHz) become additionally visible but the $n=4$ mode pair dominates.

When $\mu$ is increased to 0.46, the amplitude of the WELCH spectrum at the resonance at 2.1 kHz increases by over 10 dB/Hz (Fig. 17(a)). This resonance frequency occurs close to the $n=3$ disc mode (at $\approx 2.5$ kHz), the tangential pad mode ($P_t \approx 2$ kHz) and the radial pad mode ($P_r \approx 2.1$ kHz) identified in the response spectrum (Fig. 4) or the dissipated energy spectrum (Fig. 5).

“Periodic intermittent” [39] bursts owing to partial pad lift-offs have been observed experimentally in transient self-excited vibrations [40] and also in numerical/experimental investigations of a beam-on-beam setup [21]. Low-dimensional possibly chaotic dynamics of structural vibrations were experimentally observed by Wernitz and Hoffmann [33] during steady sliding conditions in friction brakes. Periodic bursts of radial pad vibrations were also observed in an analytical model.

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**Fig. 16.** Spectrograms in radial ($U_3^r$), tangential ($U_3^t$) and out-of-plane ($U_3^z$) displacements for friction coefficients (a)-(c) $\mu=0.05$, (d)-(f) 0.30, (g)-(i) 0.42 and (j)-(l) $\mu=0.46$; $n=4$ and $n=5$ refer to unstable split modes (0, $n=4$, 0, 0) and (0, $n=5$, 0, 0) and $P_t$ and $P_r$ to tangential and radial pad modes (Fig. 1(b)).

**Fig. 17.** (a) WELCH spectrum of $U_3^z$ for friction coefficients $\mu=0.05$ to $\mu=0.46$; (b) magnification of time trace and intervals $I_1$ and $I_2$ of laminar and chaotic motion, respectively; (c) spectrograms for intervals $I_1$ and $I_2$; and (d) power spectral densities highlighting slope in low-frequency range benchmarked with $1/f^n$ noise.
is known to be related to a subcritical ANDRONOV–HOPF bifurcation \cite{36,43}. Periodic or quasi-periodic laminar times would be observed in the Pad–Disc interface dynamics. However, according to Ben-Mizrachi et al.\cite{46}, the 1/f\textsuperscript{–}noise criterion works only with instabilities of intermittency type II or III. Further, dynamic instabilities in brake squeal have been related to ANDRONOV–HOPF bifurcations \cite{47}. Type II intermittency is known to be related to a subcritical ANDRONOV–HOPF bifurcation \cite{43}: a stable fixed point becomes unstable by colliding with an unstable limit cycle \cite{42}. Subcritical ANDRONOV–HOPF bifurcations have been found to be related to friction induced instabilities in braking systems \cite{48}. Hence the following analysis is conducted to determine if the results here are indicative of type II intermittency.

Fig. 18(a) depicts the projection of the three-dimensional phase space of the joint interval of time traces at measurement point \(3\) in \(I_1\) and \(I_2\) (Fig. 17(b)). The laminar region is plotted in ‘grey scale’ which varies from bright grey (early events) to black (latest events). The trajectories spiral out from the “time trace alternating around its mean value” of 32 \(\mu\text{m}\), which is indicated by a spiral schematically sketched in Fig. 18(a). The spiralling pattern is then interrupted by a burst of intermittent behaviour corresponding to \(I_2\) (Fig. 17(b)). However, even though the time of the burst and the amplitude seem to be random, the deterministic features become apparent when plotting the return map of the intermittent dynamics as depicted in Fig. 18(b). The identity function and a least-square fitted curve (dashed line) indicate an attractor similar to a parabola. A magnification of one small area of the return map shows one-dimensional fractality, similar to that of a coastline.

Between the identity function (the diagonal line) and the return map in Fig. 18(b)) is a channel beginning at \(\epsilon = 113\ \mu\text{m}\). When the trajectory moves into the re-injection zone (triangle in the top right hand corner of Fig. 18(b)), the motion is randomly re-injected due to random (or high dimensional) drifts in the control parameter (friction coefficient) \cite{32,42}. The value of \(\epsilon\) gives the shift from \(\epsilon\)-tangency and is proportional to the average time between chaotic bursts, that is, how long it takes, on average, to traverse the tunnel between the identity function and the attractor (half of parabola, Fig. 18(b)) \cite{42}. In Fig. 19, a histogram of the laminar time lengths, \(t_L\), in the time series of 1.8 s, is plotted and shown to be proportional to the decaying function \(t_L^{-2}\). It is possible to see a power law behaviour according to the intermittency type II route to chaos. Intermittency type II consists of quasi-periodic (predictable) laminar region which exhibits bursts of chaos (unpredictable). If the dynamics was periodic or quasi-periodic, less random laminar times would be observed. The noise floor and the slope are characteristic of type II intermittent behaviour including a low frequency modulation.

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Fig. 18. Route to chaos over intermittency type II: (a) phase-space plot shows spiralling out trajectory which belong to laminar region of interval \(I_1\) (Fig. 17); (b) intermittent bursts after reaching maximum re-injection into laminar region; (b) return map \(X\) with delay of 3.6 ms creates intermittent data; dashed line shows function fitted by means of least-square fit; shift from \(\epsilon\)-tangency \(\epsilon \approx 1.1 \times 10^{-4}\) m.
It can be summarised that the intermittent time trace, the spiralling out of the phase space, together with the slope of the PSD and the exponential decay of the histogram of laminar times as well as the fractal return map are clear evidence of the intermittency type II route to chaos [49].

For the return map in Fig. 18, the Kaplan–Yorke dimension [32] and the maximal Lyapunov exponent were calculated using the algorithm of Sano and Sawada [32]. The Kaplan–Yorke dimension is $D_{KY} = 1.089$ and the Lyapunov exponents are $\lambda_i \in [0.09, -0.98]$ for the map presented in Fig. 18. A positive Lyapunov exponent indicates chaotic motion [32], as the dynamics stretches in this dimension and has to contract in the other in order to fill out a prescribed dimension $D \leq D_{KY} \leq D + 1$, $D$ being the topological dimension and $D + 1$ the minimum dimension necessary to house the attractor.

Fig. 20 displays the operating deflection shapes of $U_z$ for $\mu = 0.46$. The intermittent pad motion is suppressed in order to examine which disc mode according to instantaneous mode squeal [6] would get excited through the pad’s motion. The rotation of the disc is indicated by an arrow around some circled elements. Below 0.3 s, no dominant mode shape can be identified (Fig. 20(a)). Then, as time increases to 0.6 s, the $n=3$ disc mode appears and then for up to 1 ms (see operating deflection shape in Fig. 20(b)). In the time interval 0.6–0.7 s, a vanishing and reappearing negative travelling wave with three nodal diameters is observed (Fig. 20(c)). After 0.7 s, this mode stabilises and starts to travel against the direction of rotation (indicated by arrows). At 0.745 s, the wave stands for a fraction of 5 ms corresponding to a peak in $U_z$, Fig. 6(c) and (f). After 0.745 s, the mode travels with the rotating disc as also observed using acoustic holography [50]. The PSD in Fig. 8 shows also a peak at 2.22 kHz ($P_1$ pad mode) which is relatively close to the $n = 3$ disc mode at 2567.8 Hz. Two physically connected solid vibrating structures can be seen as two sets of interacting oscillators. In the present case, the disc and pad are coupled through a friction interface and partial loss of contact may result from nonlinear interactions, which are dependent on the contact pressure, relative velocity and friction coefficient. As a result, connected master and slave systems
may have different dynamic attracting sets (see e.g. [36,51]). Since the stiffness of the contact changes during the application of pressure, the local friction characteristics also alter (intermittent dynamics in the contact), hence the pad modes might fluctuate over the frequency range and would be able to feed energy and interact with various modes of the disc to increase disc vibrations. This instability in the disc vibration is triggered by pad-mode instabilities and displays characteristics of the instantaneous mode squeal observed experimentally by Chen [6].

A conclusive overview of the dynamic behaviour for the pad and the disc is summarised in Table 1 with the pad dynamics being dominated by the dynamics of the pad-interface. The pad and disc are symbolised by ■ and ◯. Pad vibrations are mostly at least as unstable as the disc vibrations. Chaotic pad vibrations have a destabilising effect on the out-of-plane disc vibrations which contribute strongly to the radiated noise of brake systems [6]. Here, in-plane tangential and radial components of the pad are important. For intermittently chaotic pad motion, the disc motion intermittently chaotic pad motion, the disc motion behaves quasi-periodically. For turbulent pad motion, the disc’s out-of-plane motion shows chaotic behaviour on a torus. Pad vibrations at around 2.22 kHz have been shown to excite a dominant \( n = 3 \) out-of-plane disc mode (with resonance at 2.57 kHz). These results are similar to those found by frequency analysis for a pad-on-plate system [26] and reveal the noise generating mechanism for experimentally observed instantaneous mode squeal [6].

5. Discussions

Our recent experimental research shows that nonlinearity plays an important role in brake squeal which may result from the Ruelle–Takens route to chaos [18,19]. Supercritical Andronov–Hopf bifurcations are widely acknowledged to be the reason for squeal and are known to be a source of instability-initiating mechanisms in friction oscillators and brake systems. Results of a transient nonlinear time domain analysis (TDA) show that at low pad pressures, the pad vibration follows the intermittency type II route to chaos [32,42], which is related to subcritical Andronov–Hopf bifurcations, and triggers an unstable motion of the disc, which is the main sound radiator. This is in agreement with our latest experimental results that the dynamics of the pad vibration is more complex than that of the recorded squeal [5]. Here the pad and disc dynamics has been found to be different and this may be due to nonlinear interactions. These findings are consistent with those of Wernitz and Hoffmann [33] who found experimentally that intermittency and multiscale behaviour is the dominant dynamics in friction brake vibration. Although some unstable vibration modes of the pad are not predicted as unstable by the CEA, they are identified by the negative dissipated energy spectrum obtained by the DSA and validated by the TDA as causing self-excited vibrations. Similar pad modes with negative dissipated energy have also been identified in a numerical pad-on-plate model by Oberst and Lai [26] similar to the experimental observations of Chen [6].

The intermittency route to chaos is rarely observed as it is a transitional regime between different dynamically stable attractors (see e.g. [46]) and it cannot be detected by standard methods such as the CEA. Partial loss of contact between the pad and the disc caused by nonlinear interactions may be responsible for their different dynamic behaviour observed here (see [36,51]).

6. Conclusions

A simple model of a brake system in the form of an isotropic pad-on-disc model has been used to study the role of interactions between the pad and the disc in squeal. Although two out-of-plane disc bending modes, \( n = 4 \) and 5, have been predicted by the complex eigenvalue analysis (CEA) as unstable, all the pad modes have been predicted to be stable for the friction coefficient range from 0.05 to 0.68. By conducting a direct, steady-state analysis (DSA) of the pad-on-disc model subjected to harmonic pressure excitation, the dissipated system energy has been found to be negative not only for the two unstable disc modes but also for the two in-plane pad modes (radial and tangential) as well as for the out-of-plane pad mode, implying provision of energy by friction.

By conducting a nonlinear time domain analysis of the brake model, in-plane radial and tangential pad mode instabilities have been shown to trigger out-of-plane disc vibrations: after a periodically bursting regime, intermittent bursts of very
high amplitude appear via intermittency route type II to chaos and are related to subcritical ANDRONOV–HOPF–bifurcations. It is worthwhile to explore how much the nonlinear interactions between the pad and the disc are influenced by the real geometry, material properties and friction laws.

While the brake system's real geometry and other parts such as the calliper are important in squeal generation, this study highlights that it is not only necessary to consider a nonlinear analysis but also to consider the topology of the underlying dynamics to calculate invariant measures (fractal dimension, LYAPUNOV exponents) based on invariant sets (attractors). The analysis and understanding of the frictional contact in triggering unstable vibrations of the pad and the disc is essential in developing squeal models with better prediction of squeal propensity.

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