Design strategies for the creation of aperiodic nonchaotic attractors

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Parametric modulation in nonlinear dynamical systems can give rise to attractors on which the dynamics is aperiodic and nonchaotic, namely, with largest Lyapunov exponent being nonpositive. We describe a procedure for creating such attractors by using random modulation or pseudorandom binary sequences with arbitrarily long recurrence times. As a consequence the attractors are geometrically fractal and the motion is aperiodic on experimentally accessible time scales. A practical realization of such attractors is demonstrated in an experiment using electronic circuits.

Generating dynamics, which is aperiodic but nevertheless stable in a sense that nearby trajectories coalesce and synchronize, has been of considerable interest in the last few years. Such motion has been typically observed in driven dynamical system, in particular, when the drive is quasiperiodic. Since quasiperiodicity is difficult to achieve in practice, a major concern in this regard has been whether such dynamics can be achieved by other techniques and has been answered with various degrees of success. Here we present a simple design scheme that uses pseudorandom binary sequences with very long recurrence times to switch the dynamics between two different states. The resultant dynamics goes to an attractor, which is aperiodic and stable, namely, has negative Lyapunov exponent. Characterization of such dynamics reveals the fractal nature of such dynamics and also their differences with the ones obtained by quasiperiodic drive. Such a design scheme is further realized in an experimental setup using electronic circuits, suggesting potential applications in practical situations.

I. INTRODUCTION

The design of dynamical systems in which the motion is both aperiodic and stable has been an objective of a number of recent studies.1 One realization of this goal is in strange nonchaotic attractors (SNAs) that can be created in quasiperiodically driven systems.1,2 In such systems which have been known for over 20 yr now,3 the attractor has a fractal geometry, and this results in the dynamics being aperiodic. The Lyapunov exponents of the drive system are nonpositive: this leads to a lack of sensitivity to initial conditions and thus to the synchronization of orbits.4

Achieving quasiperiodicity is simple in principle but difficult in practice. Quasiperiodic modulation requires that the system be driven either through a single source that has an irrational frequency (for maps) or with two sources whose frequencies are incommensurate (for flows). Experimental uncertainties are usually larger than the precision with which rational or irrational numbers can be measured, and therefore finding or creating SNAs in practical situations has been proven to be difficult, except for isolated experiments.5 If strange nonchaotic dynamics is to be taken seriously, it is necessary to ask if such behavior can arise in the absence of strict quasiperiodicity. In particular, most physical, biological, and engineering systems will not be quasiperiodic, so it is natural to ask whether SNAs can arise in situations where the underlying system is autonomous or has other time dependence.

Earlier studies have addressed this issue with varying degrees of success. The use of an external noise source typically has the effect of smearing out attractor structure and gives effective Lyapunov exponents that are non-negative. As a result the dynamics is neither truly strange nor truly nonchaotic.6 The recent suggestion by Wang et al.7 that additive noise alone can be used to induce robust SNAs in both maps and flows appears to ensure that the attractors so created share the mathematical properties of SNAs formed by other bifurcation routes.8

The approach taken in the present paper pursues a different route. We show that by using parametric modulation based on deterministic pseudorandom dynamics, it is possible to create dynamical attractors that are stable, nonchaotic, and aperiodic, and which will appear strange on any measurable time scale. In this context the term “strangeness” refers purely to the geometry, while stability refers to the fidelity in signal reproduction or tracking,9 an issue that, for instance, underlies schemes for encryption and communication.

Our procedure for the creation of such aperiodic nonchaotic attractors (ANAs) relies on the use of binary random numbers to switch the dynamical system between two dynamical states, a stable fixed point and a chaotic attractor. By suitably designing the drive dynamics, it is possible to ensure that the asymptotic dynamics of the driven system has (a) a negative largest Lyapunov exponent and (b) nontrivial and complicated geometry on spatial scales that are determined by the (essentially) experimental resolution. We show that such attractors can be created in both discrete-time autono-
mous maps as well as flows and further present an experiment based on electronic circuits to support our findings. This suggests that ANAs could have potential application in practical situations where aperiodic dynamics is desirable, as for example in chaotic communications. At the same time, these attractors have differences from those created by quasi-periodic driving and other methods.\textsuperscript{1,2} It should be noted that dichotomous driving has been used before in both numerical\textsuperscript{12} as well as experimental\textsuperscript{13} studies, although the motivation there was to study noise-induced transitions between different states or attractors.

In Sec. II we discuss the design of ANAs in the driven Hénon map and the driven Lorenz system using a deterministic feedback shift register to generate a pseudorandom drive.\textsuperscript{3} In Sec. III we discuss similar driving mechanisms using chaotic sequence from a Chua circuit. The study of such attractors and their proper characterization is discussed in Sec. IV, where we show the geometric differences between ANAs and SNAs. In Sec. V we present an experimental realization of such attractors using electronic circuits and conclude in Sec. VI with a discussion and summary of our results.

II. DICHOtomous DETERministic MODULATION

Consider a dynamical system (with one freedom and a single parameter for simplicity but with obvious extension to higher dimensions and to the case of several parameters)

\[ x \rightarrow f(x, b), \]

which is modulated through the output of a binary drive sequence (strings of 0s and 1s) \( z_n \)

\[ x_{n+1} = f(x_n, b_1 + z_n(b_2 - b_1)). \]

Depending on the value of \( z_n \), the system parameters thus switch between \( b_1 \) and \( b_2 \), giving a dichotomous modulation that is, nevertheless, deterministic.

One standard way of achieving this is to use a linear feedback shift register (LFSR) (Ref. 14) that generates a pseudorandom bit sequence \( \{z\} \) through a delay mapping of the general form

\[ z_{n+1} = \sum_{i=1}^{N} a_i z_{n+1-i} \mod 2, \]

where \( a \) is also a binary variable. For a specific choice of nonzero \( a \) for a given \( N \) (the “tap sequence”), the dynamics is on an attractor with period \( \leq 2^N - 1 \). The analog generalization [namely, the analog feedback shift registers (AFSRs) (Ref. 15)] uses the same coefficients in a continuous mapping

\[ \dot{z}_{n+1} = \frac{1}{2} - \frac{1}{2} \cos \sum_{i=1}^{N} a_i z_{n+1-i} \]

to generate a pseudorandom sequence of 0s and 1s. This dichotomous drive is importantly a dynamical system and the drive sequence is an attractor of the dynamics. The sequence is optimal if arbitrarily long sequences of either 0 or 1 occur. The theory of LFSRs (and thus of AFSRs) is well developed and minimal tap sequences that produce the longest possible (namely, \( 2^N - 1 \)) period pseudorandom sequences are easily available.\textsuperscript{14} For sufficiently large \( N \), the period of the pseudorandom sequence can quickly exceed the age of the universe at any realistic sampling rate.

Designing aperiodic but nonchaotic dynamics in \( x \) is straightforward: for instance if \( b_1 \) corresponds to a case of, say, superstable dynamics, and \( b_2 \) to the case of chaotic dynamics in the system [Eq. (2)], the resultant dynamics in the driven system will be aperiodic but will rapidly be attracted to the superstable orbit whenever there is a “gap,” namely, a string of 0s. The Lyapunov exponent will consequently be negative and as a result trajectories with arbitrary initial conditions will synchronize.

Special attention should be given when choosing the tap sequence \( N \). It must be large enough such that the recurrence time, namely, \( 2^N - 1 \), is much longer than the time scales used for simulations. For smaller \( N \), the recurrence also becomes short, and since the AFSR dynamics is periodic, the system dynamics will go to a periodic attractor.

Any other random sequence will also serve the purpose, but LFSRs or AFSRs offer a practical advantage over other pseudorandom number generators (PRNGs). The shift registers are maximally stable,\textsuperscript{15} being attractors of the dynamics their stability and controllability—unlike that of stochastic sequences or PRNGs—are more easily ensured. Furthermore, since feedback shift registers are dynamical systems as well, the entire drive-response system can be represented as a delay dynamical system.

We discuss representative examples below.

A. Hénon map

The Hénon\textsuperscript{16} map is a well studied two dimensional iterative dynamical system given by the following equations:

\[ x_{n+1} = 1 + \alpha y^2_n + y_n, \]
\[ y_{n+1} = \beta x_n. \]

For a certain choice of \((\alpha, \beta)\) and for a certain range of initial condition, the system exhibits a chaotic or fixed point or periodic behavior [see Fig. 1(a)].
Applying the strategy discussed above produces attractors which are both nonchaotic and essentially strange. In the Hénon map, at $\alpha=-1.2$ the dynamics is chaotic and at $\alpha=-0.14$ the dynamics goes to a fixed point [see Fig. 1(a)]. Combining Eqs. (4)–(6), we get the following equations for the drive-response system:

$$x_{n+1} = 1 + \alpha(1 + cz_n)x_n^2 + y_n,$$  
(7)

$$y_{n+1} = \beta x_n,$$  
(8)

$$z_{n+1} = \frac{1}{2}\left[1 - \cos\left(\pi \sum_{i=1}^{N} a_n x_{n+1-i}\right)\right].$$  
(9)

If we choose $\alpha=-0.14$, $\beta=0.3$, and $c=\frac{53}{7}$, then depending on whether $z_n$ is 0 or 1, the quantity $\alpha(1+cz_n)$ will take values of either $-0.14$ or $-1.2$. As a result the dynamics hops between two different states such that the global dynamics is stable and nonchaotic. The Lyapunov exponent for the global dynamics is given by $\lambda=-0.16$. Figure 1(b) shows the attractor in phase space. Clearly the attractor looks geometrically strange; the dynamics is aperiodic and nonchaotic.

B. Lorenz system

The same strategy can be applied to a flow; switching the dynamics between a fixed point or limit cycle and a chaotic attractor can result in such strange dynamics in a modulated Lorenz system,\textsuperscript{17}

$$\dot{x} = \sigma(y-x),$$  
(10)

$$\dot{y} = (1 + c\zeta(t))\rho x - y - xz,$$  
(11)

$$\dot{z} = xy - \beta z,$$  
(12)

which has the parameter $\rho$ changing in a time-dependent manner through the variable $\zeta$.

$$\zeta(t) = z_n \quad n\tau \geq t \geq (n-1)\tau.$$

As in the mapping in Sec. II A above, $z_n$ is the output of an AFSR [Eq. (4)] and the switch duration $\tau$ is an additional parameter in the problem; it is the time for which a trajectory is switched into either of the states. In the present problem we have taken $\tau$ to switch into either of the states to be equal, but one can choose mismatched $\tau$.

The attractor shown in Fig. 2 is the result of the dynamics hopping between parameter values $\rho=7$ and $\rho=30$. At the latter value the attractor is chaotic with the characteristic butterfly structure about two symmetric unstable fixed points: $\{-8.79, -8.79, 29\}$ and $\{8.79, 8.79, 29\}$. For $\rho=7$ the system has two symmetrical attractive fixed points at $\{-4, -4, 6\}$ and $\{4, 4, 6\}$. When the parameter switches between the values, the dynamics alternates between the stable and the unstable fixed point dynamics, resulting in the structure visible in Fig. 2.

![FIG. 2. Aperiodic nonchaotic attractor obtained in the phase space for the Lorenz system under the AFSR modulation. Dynamics obtained for parameter values $\rho=7$ and $\epsilon=\frac{12}{7}$ such that $\rho$ switches between values $7$ and $30$. Switch duration time is chosen to be $\tau=10$ and the largest Lyapunov exponent is $\lambda_1=0.00985$.](image)

![FIG. 3. (Color online) (a) The Chua double-scroll attractor in the $x$-$z$ plane. The fixed parameter values are $c_1=9$, $c_2=0$, $m_0=\frac{1}{\tau}$, and $m_1=\frac{1}{\tau}$. The double scroll is obtained by tuning the parameter $c_3$. Here we take $c_3=14.141$. (b) Corresponding bit sequence obtained assigning all points with positive value of $x$ as 1 and with negative values as 0.](image)
IV. CHARACTERIZATION OF ANAs

Computation of the largest Lyapunov exponent shows that these attractors are nonchaotic. However, since the attractors are geometrically strange only over finite resolution, they differ in the qualitative and quantitative aspects of their local fluctuation properties from other similar attractors such as SNAs.

Finite-time Lyapunov exponents (FTLEs) (Ref. 20) are local estimates for the rate of divergence between nearby trajectories and explicitly depend both on the time interval $\tau$ over which they are measured as well as the initial conditions. By computing $\lambda_z$ for a large number of initial points in the phase space, one can obtain the stationary distribution

\begin{equation}
\lambda_z = \frac{1}{N} \sum_{i=1}^{N} \ln \left| \frac{\partial x_i}{\partial x_i} \right|
\end{equation}

at irrational frequency $\Omega$, the golden mean ratio $(\sqrt{5}-1)/2$, the scaling relation $|X(\Omega,N)|^2 \sim N^\beta$ is observed. For noisy motion, $\beta=1$, and the spectrum is continuous. For periodic motion, $\beta=2$, and the spectrum is discrete. For singular continuous spectrum [as in SNAs (Ref. 23)] the scaling exponent satisfies $1 < \beta < 2$. Here we find that the Fourier sum obeys scaling with an exponent slightly greater than unity; see Figs. 6(a) and 6(b). This implies that the dynamics in these attractors is typically noisy unlike SNAs, where the dynamical correlation persists over long times due to intermittency. For nonchaotic attractors in the Hénon and the Lorenz systems, we find the exponents $\beta=1.05$ and $\beta=1.04$, respectively.
A detailed characterization of the nature of dynamics can be obtained from measures based on recurrences. Recurrence plots (RPs) are defined for a given trajectory \( \{ \mathbf{x}_i \}_{i=1}^N \) through the matrix

\[
R_{ij} = \Theta(\delta - \| \mathbf{x}_i - \mathbf{x}_j \|), \quad i,j = 1, \ldots, N, \tag{23}
\]

where \( \delta \) is a predefined threshold, \( \Theta(\cdot) \) is the Heaviside function, and \( \| \cdot \| \) is the maximum norm. The maximum norm (also called infinity norm) of a vector \( \mathbf{x} \) of length \( N \) is given by \( \| \mathbf{x} \| = \max(|x_1|, \ldots, |x_N|) \). Points that are closer (respectively farther) than \( \delta \) yield an entry “1” (respectively “0”) in the matrix \( R_{ij} \). Then, values 1 and 0 are depicted as black and white dot in a two-dimensional plot, providing a visual representation of the system dynamics. The RPs exhibit characteristic large scale and small scale patterns (called typography and texture, respectively); these have been comprehensively reviewed recently. The selection criteria for the threshold \( \delta \) are discussed in details in this review by Marwan et al. Here we take \( \delta = 0.3 \sigma \) where \( \sigma \) is the standard deviation of the trajectory.

In Fig. 7, we compare the RPs for a SNA in the quasiperiodically forced Hénon map and the ANA for the modulated Hénon map. It is clear that the RP of ANA [Figs. 7(b) and 7(d)] consists of more isolated correlated points and short diagonal segments depicting short-range correlations. On the other hand, the RP for SNA [Figs. 7(a) and 7(c)] has a larger distribution of longer diagonal line segments, implying that correlation persists over long times—a signature of quasiperiodic driving.

**V. EXPERIMENT**

In the electronic experiments reported here we have designed a circuit that essentially obeys the Lorenz equations and permits one of the parameters to be switched between two values.

**FIG. 6.** (Color online) Finite-time Fourier power \(|X(\Omega, N)|^2\) vs \(T\) on a logarithmic scale for (a) the Hénon map and (b) the Lorenz system. Fractal walk of the spectral trajectory in the complex plane (Re \(X\), Im \(X\)) for (c) the Hénon map and (d) for the Lorenz system. Parameter values are as in Fig. 1(a) and Fig. 2.

**FIG. 7.** Comparison of RPs for (a) a SNA for the quasiperiodically forced Hénon map (Ref. 27) and (b) the ANA for the driven Hénon map with parameters as given in Fig. 1(b). The RP for SNA has longer diagonal line segments due to long range correlations as compared to that of the ANA, which consists of more isolated correlated points and shorter diagonal line segments. This is seen in (c) and (d) where the diagonal line distributions (with minimum length \(L_{\text{min}}\)=2 in each case) for the two cases are compared. We calculated the determinism (det) for both cases [see Eq. 46 in Marwan et al. (Ref. 25)] and for SNA, det=0.78, whereas for ANA it is det=0.62. Here we take \( \delta=0.3\sigma \) where \( \sigma \) is the standard deviation of the trajectory.

The typical parameters for the butterfly attractor are \( \sigma=10 \), \( b=8/3 \) and in our circuit the value of \( \rho \) may be switched between 28 and 7 (as in Fig. 4). The switching of \( \rho \) is controlled by a chaotic pulse generated from a Chua circuit.\(^{28,29}\) \( \rho=28 \) if the drive signal has value 1, else \( \rho=7 \) (and the drive signal has value 0). The experimental circuit of the pulse driven Lorenz oscillator is shown in Fig. 8. A Chua circuit is designed using two op-amps (U1 and U2: \( \mu \)741), capacitors C1 and C2, inductor L1 with a leakage resistance R8, and other resistances R1–R7. It generates a chaotic double scroll for choice of components noted in the circuit diagram. The dynamics of the Chua circuit can be controlled by varying R1 resistance keeping other components fixed. The double-scroll chaotic dynamics from the Chua circuit is then applied to a Schmitt trigger circuit designed by using an op-amp U3, an inverting amplifier U4, and associated resistances R9–R13. The outputs from U3 and U4 are used to control the analog switches U5A and U6A, respectively, to allow continuity of either R16 or R17 in the Lorenz circuit. The Lorenz circuit is implemented using two analog multipliers U7 and U8 and three op-amps (U9–U11: \( \mu \)741), capacitors C3–C5, and resistances R14–R21. The choice of resistances R16 and R17 made the selection of \( \rho \)-value between \( 7(=R19/R16) \) and \( 28(=R19/R17) \), respectively. The other parameters of the Lorenz circuit are decided as \( \sigma=R19/R14 \) and \( b=R19/R21 \). The analog switches are in ON state if their control pulse at voltage control point (VC) terminal is positive. So the analog switch U5A is in on state when the output of U3 is positive but U6A is in OFF state.

Alternately, the analog switch U6A is in ON state and U5A is in off state when the output of U3 is negative but inverted by the U4 to make the control pulse positive at the VC terminal of U6A. The oscilloscope picture of the control pulse as generated from the Chua circuit is shown in Fig. 9.
The upper trace is the double scroll chaotic signal from the Chua circuit, which is processed by the Schmitt trigger U3. The chaotic pulse is clearly seen in the lower trace as switching between a positive and a negative value almost randomly; the signal is scaled down in the oscilloscope.

The chaotic control signal switches the $p$-value of the Lorenz circuit aperiodically. The phase portrait of the Lorenz circuit is shown in Fig. 10: this is the ANA that results from the theoretical strategy outlined in Sec. III; see Fig. 4.

To show that the attractors obtained experimentally satisfy the synchronization condition, we constructed an auxiliary system.30 Two identical Lorenz oscillators are controlled by the chaotic pulse generated from a single Chua oscillator. The components of the Lorenz oscillators are carefully chosen with 1% tolerance so that both the oscillators are almost identical. Synchronization of two nonchaotic Lorenz circuits

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is now investigated, the circuit scheme of which is shown in Fig. 11. We observe that the output voltages of the Lorenz oscillators, OS-1 and OS-2, are completely synchronized. The oscilloscope pictures of the two time series from the oscillators are shown in upper and lower traces in Fig. 12 for comparison. The time series plotted one against the other, as shown as a thick line, confirms complete synchronization of the two oscillators (within experimental bounds). The width of the synchronization manifold is due to natural parameter mismatch between the two designed Lorenz oscillators; this is unavoidable in experiments (Fig. 13).

VI. DISCUSSION

Motion that is both stable and aperiodic is ubiquitous in natural systems. The manner in which such dynamics can be created is therefore of interest. One class of attractors that have these features has been known for some time now, but a quasiperiodic drive is essential for their creation, and thus these appear to be somewhat exceptional. An area where these considerations are potentially important is in the dynamics of biological systems. Although not manifestly periodic, several biological phenomena are stable at least in a homeostatic sense. Thus it is a moot question whether aperiodic but nonchaotic attractors are responsible for such stability.

On SNAs there is a delicate balance between global stability: as was established by Sturman and Stark there is an unstable set embedded within the attractor. The design strategy that we have enunciated in the present work keeps this feature in mind: the scheme we have proposed here is to modulate system parameters in such a manner as to achieve global stability while ensuring local instability.

This method of dichotomous modulation creates attractors, which are nonchaotic and have a fractal geometry on experimentally accessible time scales. We have recently shown that SNAs created via quasiperiodic forcing are a manifestation of weak generalized synchronization, and that similar stable attractors can be created by chaotic forcing. The parametric modulation used in the present case retains the skew-product structure of the dynamical system. The formation of these stable attractors can in some sense be seen as an instance of generalized synchronization.

The attractors created via such parameter modulation are quite distinct from SNAs. The Gaussian nature of the ANAs. The Gaussian nature of the FTLE distribution shows that the dynamics is not intermittent. From the spectral properties it is evident that unlike SNAs, the power spectrum varies as $|X(\Omega, N)|^2 \sim N$, which occurs when the motion is random or chaotic. Our system being nonchaotic, the randomness in the motion comes from the stochastic nature of the modulation. This is confirmed by looking at the correlation properties via RPs, which shows similar behavior to that of random or chaotic dynamics.

The ANAs can be realized in an experimental setup. As an example, we construct an electronic circuit experiment wherein a Chua double scroll attractor is used to drive a Lorenz attractor. The experiment closely matches the simulation result, and by experimentally constructing an auxiliary system we demonstrate that for a given drive sequence, trajectories with different initial states synchronize rapidly on the ANAs. This demonstrates the possibility of creating—or using—such dynamics in practical applications. Furthermore such a realization shows the robustness of the proposed design scheme against external noise.

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Nandi et al.  

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