Dispersive graded entropy on computing dynamical complexity

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\textbf{HIGHLIGHTS}

- We propose a complexity-DGE using the dispersion of the phase space trajectories.
- A comparison analysis shows the effectiveness and the robustness of DGE.
- DGE can clearly distinguish HRV signals of various normal and failure heart persons.

\textbf{ABSTRACT}

We propose a phase space based statistical disorder to investigate the dynamical complexity of chaotic models. The statistical disorder is defined by introducing a grade function, inversely maps the mean dispersion of the trajectories in the phase space. We denote the associate entropy by the dispersive graded entropy (DGE). Numerical investigation shows that DGE can quantify the dynamical complexity of discrete as well as continuous chaotic systems. A comparative study is also made with the other phase space based entropy measures. Finally, the proposed measure has been applied on three types of heart rate variability (HRV) signals. The results support the clinical observations related to the dynamics of healthy and congestive hearts.

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1. Introduction

Dynamical complexity is one of the most effective measures to study the amount of structural disorder in nonlinear phenomena [1–4]. Most of the real world phenomena are complex in the sense that there exists nonlinear interaction between the variables of the systems. Dynamical complexity has been well applied to living systems to investigate the inherent structural information from the dynamics [5–7]. Several neural and biomedical signals have been investigated in various pathological states and conditions. For example, it has been observed that the healthy human heart is complex, since it has many interacting subunits to keep the whole system (heart) in operation. Compare to that, a failure or blockage heart is less complex, since it fails to interact properly with its associated subunits [5,6,8,9].

The reconstructed phase space of any nonlinear real signal provides high embedding dimension in most of the cases. Significant loses can be observed for the invariant properties in case of any low dimensional phase space and its corresponding dynamics. For the practical purpose, a topological equivalent dynamics of the nonlinear system can be approximated by

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phase space reconstruction method [10]. The phase space of a system contained asymptotic information of the long term dynamics. To investigate the dynamical complexity of a system, measure of structural disorder in the phase space is thus all effective one.

Complexity of a dynamics is generally investigated by computing the entropies in both statistical mechanics and information theory perspectives [11,12]. As per as information theory is concerned, Shannon entropy, Kolmogorov–Sinai entropy, approximate entropy (ApEn), and sample entropy (SampEn) have got better applicability in the research communities for various theory and experimental situations [13–23]. For example, the complexity is directly related to the output power and its modulations, in case of theory and experimental lasers [17,18]. Using the concept of the entropies, several phase space based complexity measures has been proposed to investigate the structural disorder in the nonlinear systems [24,25]. To measure the complexity, a probability space is defined using a process which is, in fact, the transformation of the phase space [1,26–30]. As different types of transformation can be defined on the phase space, various structural patterns can be obtained from the resultant processes. It indicates that the structural patterns in the phase space are not unique. Thus, there is no optimum measure to quantify the complexity of a nonlinear system or signals. There are several threshold based entropy measures [14,24,25,31,32], proposed to investigate the dynamical complexity of a chaotic system. In these cases, different statistical structure can be constructed to measure the disorder in the system. But the only limitation is the measures depend on the threshold [14,21,22,32,41]. Moreover, the dynamics of the system does not correlate with the respective complexity measure, in these cases [27]. In order to overcome this limitation, the weighted measure is developed in [27]. In [27], the authors proposed a different structural pattern of the phase space using the concept of weights. The weights are defined by the exponential decay of the distance metric. They have also numerically shown that the corresponding weighted entropy measure correlates to the dynamics of the system (discrete or continuous). Our proposed measure depends on the phase space and can predict the complexity of both the discrete and continuous models in any dimension. Furthermore, the dispersive graded function \( f : D \rightarrow G \subseteq [0, 1] \) (D represents the matrix of the Euclidean distances between the trajectory points) results a non-symmetric matrix which can describe the detail structure of the phase space. As grades correspond to the dispersion of the trajectory with respect to each points in the phase space, the DGE measure is thus highly sensitive to any changes in the phase space.

The article is organized as follows: In Section 2, we proposed a measure of statistical disorder, corresponds to the dispersion of the trajectories of a chaotic system in terms of their distance based grades. Based on the proposed statistical disorder, an entropy is defined. To investigate the applicability of the proposed measure, we consider the regular and chaotic dynamics of the discrete logistic map and the Lorenz system. The regular and chaotic dynamics in a deterministic system can be quantified by the method of maximal Lyapunov exponent \( (\lambda_{\text{max}}) \) [33–35]. In the case of physical and biological application, where the underlying dynamics is unknown, the method of phase space reconstruction [10,33,34,36,37] is adopted. However, there are some problems inherent in phase space reconstruction as discussed in detail in [38,39]. On the other hand, 0–1 chaos test method can detect the chaotic and non-chaotic behaviour of the system directly from the signals [40,41]. The main advantages of this method are (i) it is binary (minimizing issues of distinguishing small positive numbers from zero) and (ii) it does not suffer from the difficulties associated with phase space reconstruction. Also, it can be universally applied to any deterministic dynamical system, in particular to ordinary and partial differential equations, and to maps. A correlation analysis is then performed between the quantification measure of 0–1 test and our proposed measure. It has been observed that the measure is according to the dynamics of both the systems. We have also compared our proposed measure with the existing phase space based complexities, for the performance and compatibility of our tool. In Section 3, we have applied the DGE in case of classifications of HRV signals. It has been observed that, \( H_{\text{DGE}} \) of normal healthy person (NHP), is significantly high compare to the same for atrial fibrillation (AF) and congestive heart failure patients (CHFP). We have implemented the results with 20 different samples of each cases. The proposed tool can be successfully distinguished the three classes NHP, AF and CHFP with the proper thresholds, and justified our previous observations [8]. The tool can be implemented as a device to classify several biomedical signals and other real signals.

2. Methodology

In this section, we define a statistical disorder from a phase space. Then an entropy measure—DGE is constructed.

2.1. Dispersive graded entropy

For \( n \)-dimensional phase space \( P = \{\overrightarrow{x}_i \in R^n\}_{i=1}^N \) (\( N \) being the length of the trajectory), the distance between the two points \( \overrightarrow{x}_i, \overrightarrow{x}_j \in P \) can be defined as \( d_{ij} = \| \overrightarrow{x}_i - \overrightarrow{x}_j \| \), \( \| \| \) represents the Euclidean norm. Then, the matrix \( D = (d_{ij})_{N \times N} \) contains information of the distance between each pair \( \overrightarrow{x}_i, \overrightarrow{x}_j \in P \). We consider mean \( M_i \) of the time series \( \{d_{i1}, d_{i2}, d_{i3}, \ldots, d_{IN}\} \) for \( i = 1, 2, \ldots, N \). So, \( M_i - d_{ij} \) represents the dispersion of \( d_{ij} \) from \( M_i \). Since \( d_{ij} \) affects due to changes in the movements of the trajectories of \( P \), the variation of \( M_i - d_{ij} \) will be an effective observation to characterize the dynamics of the phase space.

We define a function \( f : \{d_{ij}\} \rightarrow \{g_{ij}\} \) by

\[
g_{ij} = f(d_{ij}) = \frac{1}{\alpha + (M_i - d_{ij})^\gamma}, \quad (\alpha \geq 1)
\]

(1)

where \( M_i \) represents mean of \( \{d_{i1}, d_{i2}, d_{i3}, \ldots, d_{iN}\} \) (\( i = 1, 2, \ldots, N \)). We denote the matrix \( (g_{ij})_{N \times N} \) by \( G \).
non-negativity of $f$ cases. It indicates that dispersion of the series $\{D\}$ matrix (distance matrix). Based on Shannon information entropy concept, we defined dispersive graded entropy (DGE) by (4).

Then, each $g_{ij} \in G$ represents a grade with respect to $d_{ij} \in D$ under the function $f$. The square of $M_i - d_{ij}$ is taken for non-negativity of $f$. The term $\alpha \geq 1$ scales the values of $d_{ij}$ to the range $[0, \frac{1}{\alpha}]$. From (1), it follows that the minimum dispersion corresponds the values $\frac{1}{\alpha}$ when $d_{ij} = M_i$. For the other cases, $f < \frac{1}{\alpha}$. Thus, $G$ can describe information about the dispersion of the trajectories in term of their respective grades. As the disorder in $P$ can be described by the graded matrix $G$ with the function $f$, we can denote it as the dispersive graded disorder under $f$. The dispersive graded space (DGS) with the function $f$ is defined by $(G, f)$.

Fig. 1a is the schematic diagram of the proposed scheme with four arbitrary points $\vec{x}_i$, $i = 1, 2, 3, 4$ on the trajectory. Since $d_{ij} = d_{ji}$, $\forall i \neq j$ and $d_{ii} = 0$, $\forall i = j$, we always get a symmetric matrix $D = (d_{ij})_{4 \times 4}$. The matrix $D$ is shown in Fig. 1b. From Fig. 1b, the distance between the points $\vec{x}_1, \vec{x}_2$ can be measured by the associate color bar. For each $i$, the corresponding row of $D$ can be considered as a univariate time series $\{d_{1i}, d_{2i}, d_{3i}, d_{4i}\}$. From Fig. 1c, it can be observed that $M_i$s are different in all cases. It indicates that dispersion of the series $\{d_{1i}, d_{2i}, d_{3i}, d_{4i}\}$ about $M_i$s can be effective to measure the structural disorder of the phase space. Using (1), we then transform all the distances $d_{ij} \in D$ into $g_{ij}$. The corresponding matrix $G = (g_{ij})_{4 \times 4}$ is given in Fig. 1d. As we have fixed $\alpha = 1$, the range of $f$ is given by $[0, 1]$, i.e., $\{g_{ij}\} \subseteq [0, 1]$. Thus dispersion of the trajectories can be measured by the matrix $G$ under the function $f$ which we denoted by $(G, f)$. For our purpose, we use $G$ to represent $(G, f)$ in all the following discussions:

Based on the grades $g_{ij} \in G$, We define an average grades $g_i$ by

$$g_i = \sum_{j=1}^{N} g_{ij}, \quad (i = 1, 2, \ldots, N)$$

(2)

Since $g_{ij}$ quantifies dispersion between the states $\vec{x}_i$ and $\vec{x}_j$, so each $g_i$ $(i = 1, 2, \ldots, N)$ can describe total dispersion over a trajectory segment with initial state $\vec{x}_i$. Which implies that, $g_i$ can characterizes the amount of statistical disorder in the system through its frequency distribution $P(g_i)$. The probability of $g_i$ is defined by

$$p(g_i) = \frac{P(g_i)}{\sum_{i=1}^{N} g_i}.$$  

(3)

Based on Shannon information entropy concept, we define dispersive graded entropy (DGE) by

$$H_{DG} = -\sum_{i=1}^{N} p(g_i) \log p(g_i).$$

(4)
2.2. Numerical verification

In this section, verification of the measure $H_{DC}$ is numerically investigated on both the discrete and continuous systems. In order to show the effectiveness of $H_{DC}$, we have also compared the results with the weighted recurrence based entropy [27].

2.2.1. Discrete system

For the discrete case, we consider the Logistic map:

$$x_{n+1} = cx_n(1-x_n), \quad x_0 = 0.1,$$

(5)

where $c$ represents the bifurcation parameter.

The bifurcation phenomena of (5) have been investigated within the range $c \in [3.5, 4]$, which is given in Fig. 2a. From the figure, it can be observed that the system has period doubling route to chaos with several periodic window. To investigate the dynamics in more detail, we have applied the $0 \rightarrow 1$ test [39,40].

In $0 \rightarrow 1$ method, a signal $x(n)$ $(n = 1, 2, \ldots, N, N$ being the length of the signal) is translated by

$$p(n, \nu) = \sum_{j=1}^{n} x(j) \cos(j\nu) \quad \text{and} \quad q(n, \nu) = \sum_{j=1}^{n} x(j) \sin(j\nu),$$

(6)

where $\nu \in (0, \pi)$.

The $(p, q)$ cloud takes an important role to quantify the regular as well as the chaotic behaviour of a system. In fact, regular and random walk or Brownian motion like pattern can be observed in Fig. 2c. It signifies chaotic behaviour of the system. From Fig. 2c, it can be observed that the geometrical pattern of the $pq$-plot has a regular structure. It indicates regular dynamics of (5) at $c = 3.63$. On the other hand, random walk or Brownian motion like pattern can be observed in Fig. 2c. It signifies chaotic dynamics in (5) at $c = 3.63$. In order to investigate the nature of the corresponding dispersive graded space of (5), we have analyzed the respective dispersive graded matrix $(g_{ij})$. Fig. 2d and e shows $(g_{ij})$ matrix plots of the respective phase spaces of (5) for $c = 3.63$ and 3.9 respectively. From Fig. 2d, small variation in the colors of the matrix plot can be observed. It implies repetitive patterns of the grades in the corresponding $(g_{ij})_{N \times N}$. As repetitive patterns indicates that the variations in dispersive grade are small in numbers, so nearly periodic pattern can be observed in the dispersions between the trajectory points. It corresponds regular phase space as well as regular dynamics of the system (5). On the other hand, Fig. 2e shows large number of variation in the colors. It implies variable patterns of the grades in the corresponding matrix plot. As variable grades correspond to the variable distances between the trajectory points in the phase space, so patterns of the respective dispersion are very irregular. It signifies irregular structure of the phase space. Thus, regular as well as irregular structural patterns can be characterized by $(g_{ij})_{N \times N}$. Moreover, it indicates that the dynamics of (5) can be recognized using $(g_{ij})_{N \times N}$ for all $c \in [3.5, 4]$.

The diffusive and the non-diffusive behaviour of the system can be characterized from the above $pq$-plots using $M^n_x$ [40], given by

$$M^n_x = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} [p(j + n, \nu) - p(j, \nu)]^2 + [q(j + n, \nu) - q(j, \nu)]^2,$$

(7)

where $n \leq n_{cut}$ and $n_{cut} \ll N$. In practice, the value of $n_{cut}$ is taken by $n_{cut} = \frac{N}{10}$.

In [37], it can be also observed that, the asymptotic growth of the function $M^n_x$ can quantify regular as well as chaotic dynamics. The asymptotic growth ($K_v$) is defined as

$$K_v = \lim_{n \to \infty} \frac{\log M^n_x}{\log n}.$$

(8)

The values of $K_v \approx 1$ and 0 indicates chaotic and regular dynamics of the system [40].

We have calculated $K_v$ for the system (5) with $c \in [3.5, 4]$, which is given in Fig. 3a. From Fig. 3a, it can be observed that the values of $K_v$ at $c = 3.63, 3.9$ are nearly 0 and 1 respectively. It corresponds the regular and chaotic dynamics at $c = 3.63, 3.9$ respectively. So, $K_v$ quantifies the dynamics of the system. It can also be observed that the fluctuation of $K_v$ has a strong correlation with the bifurcation diagram (shown in Fig. 2a). So, dynamics of (1) can be quantified by investigating the fluctuations of $K_v$ with $c \in [3.5, 4]$.

We next compare that dynamics with the measures $S_{WRP}$ (given in [29]) and $H_{DC}$ over $c \in [3.5, 4]$. To compare the correlations, we follow the fluctuation of $K_v$ over $c \in [3.5, 4]$. The measure $S_{WRP}$ is based on weighted recurrence [24], defined as

$$S_{WRP} = - \sum_{(s_i)} p(s_i) \log p(s_i),$$

(9)

where $s_i$ is given by $s_i = \sum_{j=1}^{N} W_{ij}$. $W_{ij} = e^{-|x_i - x_j|}$ represents weighted recurrence between two points $x_i, x_j$ in the phase space. $p(s_i)$ is the probability of the occurrence $s_i$ in the set $(s_1, s_2, \ldots)$. Using (4) and (9), we have investigated $S_{WRP}, H_{DC}$
with $c \in [3.5, 4]$ for the system (5). The corresponding fluctuations are given in Fig. 3b, c respectively. From Fig. 3b, c and Fig. 3a, similar trends in the respective fluctuations can be observed. Similar investigation has been done for the continuous system.

### 2.2.2. Continuous system

For the continuous case, we have considered the Lorenz system given by

$$
\dot{x} = \sigma(y - x), \quad \dot{y} = x(\rho - z) - y, \quad \dot{z} = xy - \beta z.
$$

where $\sigma$, $\rho$ and $\beta$ are the system parameters. In our case, we fixed $\sigma = 10$, $\rho = 28$. The initial conditions are taken as $x(0) = 8$, $y(0) = 9$, $z(0) = 25$.

To understand the behaviour of system (10) with respect to variable $\beta \in [\frac{1}{3}, \frac{8}{3}]$, we first construct the bifurcation diagram, shown in Fig. 4a.

From the figure, similar phenomena can be observed as seen in Fig. 2a. Using (6), we have investigated the $pq$-plots of (10) at $\beta = 1$, $\frac{2.5}{3}$ respectively. To compute $p(m, v)$, $q(n, v)$, we first calculated Poincare’ section from the phase space of (10). Then, any one component of the Poincare’ sections is considered as $x(j)$ for calculating (6). The corresponding $pq$-plots are shown in Fig. 4b and c respectively. From Fig. 4b, it can be observed that the geometrical structure of the $pq$-plot is regular. It indicates regular dynamics of (10) at $\beta = 1$. On the contrary, random walk like structure can be seen in the same at $\beta = \frac{2.5}{3}$ (see Fig. 4c). It implies that the corresponding dynamics is chaotic. To investigate the respective DGS, we have calculated $(g_{ij})$ of (10) with $\beta = 1$, $\frac{2.5}{3}$. Fig. 4d and e shows the corresponding $(g_{ij})$ matrix plots at $\beta = 1$ and $\frac{2.5}{3}$ respectively. From
Fig. 3. (a) represents $c$ vs. $K_\nu$ graph of (5) with $c \in [3.5, 4]$. For each $c$, $K_\nu$ is calculated using (8) by considering $n_{cut} = 1000$. The length of solutions are taken same in each computation and is equals to 10000. For $c \in [3.5, 4]$, (b), (c) represents $c$ vs. $S_{WRP}$ and $H_{DG}$ graphs respectively. In each case, probability is counted with bins = 50. In order to calculate $g_{ij}$, we have fixed $\alpha = 1$ for all cases.

the figures, similar patterns can be observed in the respective DGS as observed in Fig. 2d and e respectively. So, regular (at $\beta = 1$) as well as chaotic dynamics (at $\beta = \frac{7.5}{3}$) of (10) can be recognized using (1). Using (8), we have also quantified the respective dynamics.

We have quantified the dynamics of (10) at $\beta = [1 \frac{7.5}{3}]$. The values of $K_\nu$ found to be 0.0052 and 0.9986 respectively (see Fig. 5a). It corresponds the regular and chaotic dynamics at $\beta = 1$ and $\frac{7.5}{3}$ respectively and hence $K_\nu$ can successfully quantifies the respective dynamics of (10). The fluctuation of $K_\nu$ is also investigated under the variation of $\beta \in [\frac{1}{2}, \frac{8}{3}]$. Fig. 5a shows the corresponding $\beta$ vs. $K_\nu$ graph. A strong correlation between the fluctuation and the respective bifurcation can be observed from Fig. 5a. It proves that $K_\nu$ can quantifies the dynamics of the system (10).

In the previous Section 2.2.1, similar investigation is done on the fluctuations of $S_{WRP}$ and $H_{DG}$ over $\beta \in [\frac{1}{2}, \frac{8}{3}]$. The corresponding graphs are shown in Fig. 5b, c respectively. From Fig. 5b and c, it can be seen that the fluctuations of $S_{WRP}$ and $H_{DG}$ having similar trend as that of $K_\nu$.

From the whole analysis, it implies that both the measures $S_{WRP}$ and $H_{DG}$ are effective to compute the dynamical complexity of a system, from its phase space. It is important to investigate a comparison between these two measures, in terms of performance analysis. To establish the better applicability, we have verified the correlations of both the measures with the dynamics of a discrete and continuous system. In order to investigate that, we have constructed the window cross-correlation analysis for both the measures with $K_\nu$ (as $K_\nu$ corresponds to the dynamics of the system). The, cross-correlations $r(S_{WRP}, K_\nu), r(H_{DG}, K_\nu)$ has been computed with the variable $Lag \in [-4, 4]$ and window $W_\omega (\omega = 1, 2, \ldots, 35)$ for the Logistic map as well as the Lorenz model, and the corresponding contours are shown in Fig. 6. From Fig. 6a, it can be observed that the values of $r$ decreases for some window at $Lag = 0$. It indicates, inconsistent correlation between $S_{WRP}$ and $K_\nu$ over $W_\omega$. On the other hand, a consistent high correlation between $H_{DG}$ and $K_\nu$ can be observed in Fig. 6b. Similar trend can be observed for the continuous case, given in Fig. 6c and d for the $S_{WRP}$ and $H_{DG}$ respectively. So, the analysis indicates that $H_{DG}$ is more close to the dynamics compare to $S_{WRP}$. Thus, it is established that our proposed entropy measure is more robust and has better applicability than $S_{WRP}$ to investigate the dynamical complexity of a deterministic system. We have also verified the applicability of our proposed measure in case of real application. In the following section, we have discussed the effectiveness of $H_{DG}$ on biomedical signals.

3. Application on heart rate variability signal

In this section, we have studied dynamical complexity of the HRV signals. The study is made by using $H_{DG}$ on three types of HRV signals—normal healthy persons (NHP), atrial fibrillation (AF) and congestive heart failure patients (CHFP) respectively. The signals are collected from Physionet database [42]. For the NHP, we have selected 20 normal sinus rhythm RR time series. On the other hand, same number of RR time series was taken for AF and CHF patients. In each case, the reordered ECGs are digitized at 128 samples/s.
Fig. 4. (a) represents $c$ vs. $x_{\text{max}}$ graph of the system (10) with $\beta \in [\frac{1}{3}, \frac{8}{3}]$. The red colored lines indicate the bifurcation region at $\beta = 1$ and 2.5 respectively. In order to find $x_{\text{max}}$, we have considered the local maxima of the solution $x$ satisfying (10) with length 10000. (b) and (c) represent $p_q$-plots of the solution $x$ of length 10000 at $\beta = 1$ and 2.5 respectively. In order to calculate the values of $p(m, \nu)$ and $q(m, \nu)$, we have considered Poincaré sections by $z = \bar{z}$, where $\bar{z}$ denotes the mean of $z$. The corresponding $(g_{ij})_{200 \times 200}$ matrix plots are represented in (d) and (e) respectively. In each case, $g_{ij}$ are calculated using (10) with $\alpha = 1$. The color bars indicate the values of $g_{ij}$. The lower and upper bound in the color bars represents the maximum and minimum value of the respective $(g_{ij})_{200 \times 200}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In order to studied the complexity, we have investigated the fluctuation of $H_{DG}$ over different window

$$W(i) = [RR_{1+(i-1)\text{i}}, RR_{n+(i-1)\text{i}}], \ (i = 1, 2, \ldots, M)$$

(11)

where $N$ begin the length of the RR series and $n, l, M$ satisfy the condition $M \leq \frac{N + l - n}{n} - 1$.

To calculate the value of $H_{DG}$, we considered Poincaré plot of the RR time series. The fluctuations of $H_{DG}$ over $M(= 20)$ windows are investigated by the surface fitting for the NHP, AF and CHFP. The corresponding complexity surfaces $S_{\text{NHP}}, S_{\text{AF}}, S_{\text{CHFP}}$ are given in Fig. 7a. From the figure, it can be observed that values of $S_{\text{NHP}}$ > values of $S_{\text{AF}}$ > values of $S_{\text{CHFP}}$ over all windows $W(i), i = 1, 2, \ldots, 20$. It implies exactly same inequality in their respective complexities. So, $H_{DG}$ can quantify as well as differentiate the dynamical complexity of NHP, AF and CHFP.

We next measure mean trend of the fluctuation over all windows, shown in Fig. 7b. Similar trend in the complexity can also be observed in Fig. 7b. The differences between the non-intersecting mean fluctuations are defined by

$$\Delta H_{DG} = \min\{mH_{DG}^{\text{NHP}}\} - \max\{mH_{DG}^{\text{CHFP}}\}, \ (\text{case 1})$$

$$= \min\{mH_{DG}^{\text{NHP}}\} - \max\{mH_{DG}^{\text{AF}}\}, \ (\text{case 2})$$

$$= \min\{mH_{DG}^{\text{AF}}\} - \max\{mH_{DG}^{\text{CHFP}}\}, \ (\text{case 3})$$
Fig. 5. (a) represents $\beta$ vs. $K_\nu$ graph of (10) with $\beta \in [1, \frac{7}{3}]$. For each $\beta$, $K_\nu$ is calculated using (8) by considering $n_{cut} = N_0 / 10$ ($N_0$ being the number of points in the respective Poincare’ sections). In each case, Poincare’ sections are taken as $z = \bar{z}$, where $\bar{z}$ represents statistical mean of the $z$-component. The length of the solutions are taken same in each computation and is equals to 20000. For $\beta \in [1, \frac{7}{3}]$, (b) and (c) represents $\beta$ vs. $S_{WRP}$ and $H_{DG}$ graphs respectively. In each case, probability is counted with $= 50$ bins. The graded matrices $G$ are computed by considering $\alpha = 1$ for all cases.

Fig. 6. (a), (c) represents the respective contour diagrams of $r(S_{WRP}, K_\nu)$ with $\omega = 4, 2, \ldots, 35$ for the Logistic and Lorenz systems. (b) and (d) represents the cross-correlation of $r(H_{DG}, K_\nu)$ with $\omega = 4, 2, \ldots, 35$ for the Logistic and Lorenz systems respectively. In order to compute the cross-correlation, we have considered the whole fluctuations of $K_\nu, S_{WRP}$ and $H_{DG}$ as shown in Figs. 3 and 5 respectively during the windowing process. $W_\omega$ is the $\omega$th sub-interval of the intervals $[3.5, 4]$ (for c) and $[1, \frac{7}{3}]$ (for $\beta$) respectively during the windowing process. The width of each window is fixed as 100 for all $\omega$.

where $\{m_{NHP}^{\text{DC}}, m_{AF}^{\text{DC}}, m_{CHFP}^{\text{DC}}\}$ represent the mean fluctuation of the complexity surface for NHP, AF and CHFP respectively. The numerical values of the $\Delta H_{DC}$ are also investigated. The values are found to be 2.3, 0.7, 1.2 for case 1, case 2 and
Further, it can also differentiate the patterns of the AF and CHFP with the threshold $H_{DG}$ case 3 respectively. So, $H_{DG}$ can differentiate the NHP from both AF and CHFP with the respective threshold $\Delta H_{DG} \approx 2.3, 0.7$. Further, it can also differentiate the patterns of the AF and CHFP with the threshold $\Delta H_{DG} \approx 1.2$.

In order to verify the statistical consistency of these results, we have considered null and its alternative hypothesis in each case. The whole scenario can be seen from the given box plot in Fig. 7c. It is evident from the box plot that the null hypothesis are rejected with 95% confidence level (where $p$-value $= 8.6983 \times 10^{-56}, 1.5075 \times 10^{-37}, 2.8017 \times 10^{-44} < 0.05$). So, the above statistical analysis implies that the series $H_{DG}^{NHP}, H_{DG}^{NHP}$ and $H_{DG}^{CHFP}$ comes from different populations. It also indicates less complex dynamics exist in the HRV signals of CHFP than that of AF and NHP. Moreover, this result strongly correlates with clinical and experimental biomedical observations [8,43]. Similar observation has been also made for the aforesaid HRV signals with multiscale entropy (MSE) method [5,6]. In [6], it has been observed that values of MSE for NHP and CHFP are equal at scale 2. The complexity for those then increases monotonically with respect to the scale ($>2$). However, the changes does not follow any specific trend. Moreover, the fluctuations of $S_{H}$ have been overlapped for the NHP, CHFP and AF within the range $[12,30]$. So no definite conclusion can be made from these analysis regarding the specific trend of the dynamics. On the other hand, our proposed method shows a significant difference between the complexity of NHP and CHFP over all window. In fact, a proper threshold can be found out between their complexities. Moreover, MSE is defined for the signal itself, while our proposed DGE is obtained from the Poincaré plots (or 2D phase space with lag=1) of the HRV signals. Thus, the DGE method is more robust than MSE to describe the dynamical complexity of the HRV signals.

### 4. Conclusion

We propose a type of disordered space-DGS to quantify the dynamical complexity of chaotic systems. The corresponding dispersive grade is defined by a function which maps dispersion of the trajectories into $[0, \frac{1}{2}] (|\alpha| < 1)$ and the graded matrix $G$ can recognize disorder in the phase space of a system. By considering the average grade of the points in the trajectory, a probability distribution and hence a Shannon based complexity measure DGE is then defined. The numerical results also shows a strong correlation between the fluctuation of $H_{DG}$ and $K_{Pr}$ with the bifurcation parameters for both discrete and continuous systems respectively. A comparison of $H_{DG}$ with $S_{WPR}$ is done by a window cross-correlation analysis. The analysis indicates more higher correlation of the measure $H_{DG}$ with the dynamics than the same of $S_{WPR}$. In this way, we have established the better applicability of our proposed measure. Finally, the DGE is applied on three types of HRV signals. For each signals, the fluctuations of $H_{DG}$ has been observed. Non-intersecting surfaces ensures the effectiveness of $H_{DG}$. The mean of the fluctuating surfaces show unique trend for each types of HRV signals. Statistical test also confirmed the consistency of the numerical results. Thus, method of DGE can measures the dynamical complexity as well as pattern of the dynamics of the HRV signals.

### References


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**Fig. 7.** (a) represents fluctuating surfaces of $H_{DG}$ over 20 HRV signals and window index. The upper, middle and lower surfaces represent complexity for NHP, AF and CHFP respectively. To construct Poincaré’ plot, 2D phase space of the signals with unit lag is considered. The color bar indicates values of $H_{DG}$. To compute $H_{DG}$, we have considered $\alpha = 1$ and 50 bins. (b) represents the corresponding mean trend fluctuations ($mH_{DG}^{NHP}, mH_{DG}^{NHP}, mH_{DG}^{CHFP}$) with ‘blue’, ‘green’ and ‘red’ colors respectively. The double arrow indicate their respective entropy differences. (c) represents the notched box plot for the three HRV samples-NHP, AF and CHFP. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)


