Signal detection based on recurrence matrix statistics

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Abstract

This work considers the problem of detecting signals in noise in the absence of a well-defined signal model. Specifically, we compare detectors based on recurrence plots to one of the more commonly used detection strategies. Results indicate improvements are possible using the recurrence-based detectors for certain signal-to-noise ratios.

1. Introduction

Detection of deterministic signals buried in white noise is an essential tactical requirement for present-day military platforms in order to avoid detection by hostile radar systems. However, current low-probability of intercept (LPI) radar systems spread the energy they transmit over a large spectral range, reducing the power in the signal to a level below the thermal noise in the target platform’s radar-warning receiver. An effective receiver is one that can overcome this noise floor and detect the signal with low Type-I and Type-II error.

A wide array of detection techniques exist for finding a signal in background noise. In each the general approach is the same: compute a test statistic from the data and compare it to a detection threshold. Which statistic to compute and where to set the threshold depend on the amount of a priori knowledge one has about the signal and on the costs assigned to a missed detection. For example, when detecting known signals in additive Gaussian noise an optimum receiver (test statistic) has been developed based on the likelihood ratio, yielding the Matched Filter. The detection threshold can be set using any one of a number of criteria for optimality including: maximum a posteriori (MAP), Bayes’ criterion, and Neyman–Pearson criterion to name a few. An extensive literature has also evolved to handle cases where the parameters of the signal are only partially known. A thorough review is given in McDonough and Whalen [1]. The case of interest in this paper is that in which nothing is known a priori about the parameters of the incoming signal. This is the most difficult detection scenario, and some form of power or normality detector is often regarded as the only option.

This Letter explores the utility of recurrence plots (RPs) for detecting signals hidden in noise, when the signal is completely unknown and the background noise is Gaussian. The RP was originally developed to check for stationarity in dynamical systems [2]. It has proven to be very useful in both depicting and analyzing complex, nonlinear, possibly chaotic, systems. Examples may be found in fields as diverse as economy and earth science, astrophysics and physiology (see [3] for a comprehensive review). Applications of RP techniques to signal detection have also appeared in the literature [4–6]. For example, Zbilut et al. used detectors based on RPs to distinguish deterministic signals from noise [4,6]. A detector based on cross-recurrence plots has also been used to extract signals from noise and was compared to a spectral detection scheme [5]. Recurrence plots have even been used to distinguish signal from noise in physiologically generated signals (EMG signal) in Filligoi [7]. Another promising approach to signal detection based on RPs appears in Ref. [8].

This Letter seeks to better understand the power of recurrence-based detectors as compared to more conventional approaches. The recurrence-based detector derived here focuses on the distribution of distances in an unthresholded RP (all of the information contained in a thresholded RP is contained in an unthresholded RP which is much easier to work with analytically). This distribution...
is derived analytically and is used as the basis for a chi-squared test for the presence of a signal. A second detection scheme using a chi-squared test for normality is also employed for comparison purposes.

All results are displayed using Receiver Operating Characteristic (ROC) curves, a commonly used approach for displaying the Type-I and Type-II error associated with a given receiver. Even though it has been shown [9] that the expectation of unthresholded recurrence plots can often be computed simply from second-order statistical quantities, the results presented here illustrate an advantage to using detection strategies based on RPs for certain signal-to-noise ratios.

2. Recurrence plots

Recurrence plots were originally intended as a tool for analyzing the output of nonlinear, dynamical systems and have been used to estimate a host of quantities pertaining to the dimensionality and stability of the system being observed [3]. The goal in this work, however, is to use the recurrence plot as a tool for detecting the presence of a signal in noise and not to draw inference about the system that produced the signal.

Consider the vector \( \mathbf{x} = \{x(1), x(2), \ldots, x(m), \ldots, x(M)\} \) consisting of \( M \) real data values sampled at discrete times \( t = m \Delta t \). From this single vector, a new family of \( N \) time-delay vectors of length \( n < M \) is constructed by means of a time delay embedding \( \mathbf{x}_i = \{x(i), x(i+1), \ldots, x(i+(n-1)L)\} \) for \( i = 1, 2, \ldots, N \), where \( n \) is referred to as the embedding dimension and \( L \) is a measure of time delay. Clearly \( M = N + (n-1)\). Hence the \( M \) one-dimensional data points have given rise to \( N \) vectors of dimension \( n \). Prescriptions for selecting \( L, n \) can be found in [10], however, these approaches were designed for use with the output of a deterministic dynamical system subject to relatively low levels of noise. By contrast our application involves very high noise levels, thus the standard algorithms will often fail. The approach used here, therefore, is to vary both \( L \) and \( n \) as parameters associated with the proposed detector.

The unthresholded recurrence matrix is the \( N \times N \) matrix \([d_{i,j}]\) where

\[
\begin{bmatrix}
\sqrt{67} & \sqrt{62} & \sqrt{21} & 0 \\
\sqrt{30} & \sqrt{21} & 0 & \sqrt{21} \\
\sqrt{21} & 0 & \sqrt{21} & \sqrt{62} \\
0 & \sqrt{21} & \sqrt{30} & \sqrt{67}
\end{bmatrix}
\]

is the distance between the vectors \( \mathbf{x}_i \) and \( \mathbf{x}_j \). In this Letter, this distance will always be the Euclidean distance; that is, the square root of the sums of the squares of the corresponding component differences. Other definitions of distance have been used in other applications. This unthresholded matrix will be the concept of differences. Other definitions of distance have been used in other applications. Each of these matrix terms will be interpreted as such. For the purposes of our analysis, we require the probability distribution of the length and the length squared of the vector \( \{x_1, x_2, \ldots, x_n\} \), where each \( x_i \) has probability density function

\[
p(x_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x_i^2/2\sigma^2}, \quad i = 1, 2, \ldots, n.
\]

Note that in this example the time delay vectors are all distinct, but there are clearly redundancies in the distance values (beyond those due simply to the symmetry of the matrix). Despite these redundancies we will show that detectors based on \( d_{i,j} \) possess an advantage over those based strictly on the original \( M \) data values.

3. Distribution of RP distances in the “Noise Only” case

In order to formulate the signal detection strategies to be assessed in this Letter, some basic probability distributions need to be calculated. In these derivations, \( x_1, x_2, \ldots, x_n \) are defined to be independent Gaussian random variables, each with zero mean and variance \( \sigma^2 \). The \( x_i \) used here are arbitrary random variables and are not related to the \( X_i \) defined already as the delay vectors. Additionally, the index \( n \) is not necessarily related to the embedding dimension although later it will be interpreted as such. For the purposes of our analysis, we require the probability distribution of the length and the length squared of the vector \( \{x_1, x_2, \ldots, x_n\} \), where each \( x_i \) has probability density function

\[
p(x_i) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x_i^2/2\sigma^2}, \quad i = 1, 2, \ldots, n.
\]

The formulas for these probability distributions can be found in the literature; for example, a proof by a geometrical method is found in Parzen [11] and a proof by means of characteristic functions is found in McDonough and Whalen [1]. Here we present a direct proof that, to our knowledge, has not appeared previously.

We first derive the probability density function for \( r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \). This can be achieved readily if we can obtain a one-to-one transformation on \( R^n \) in which \( r \) is one of the transformed variables and \( r \) is also separable from the other variables in the joint probability density function of the transformed variables. A transformation of variables \( \{x_1, x_2, \ldots, x_n\} \to (r, \theta_1, \ldots, \theta_{n-1}) \) satisfying these requirements is given by

\[
x_1 = r \cos \theta_1,
\]
\[
x_2 = r \sin \theta_1 \cos \theta_2,
\]
\[
x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3,
\]
\[
\vdots
\]
\[
x_{n-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1},
\]
\[
x_n = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1},
\]

where \( 0 \leq r < \infty \) and \( 0 \leq \theta_i < \pi \). Then

\[
p(r, \theta_1, \ldots, \theta_{n-1}) = J_n \prod_{i=1}^{n} p(x_i),
\]

where \( J_n \) is the Jacobian of the transformation. A proof by induction [12], which proceeds by expanding by cofactors of the first row, yields the formula

\[
J_n(r, \theta_1, \theta_2, \ldots, \theta_{n-1}) = r^{n-1}(\sin \theta_1)^{n-2}(\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2}
\]

and so

\[
p(r, \theta_1, \ldots, \theta_{n-1}) = r^{n-1}(\sin \theta_1)^{n-2}(\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2} \frac{1}{\sigma \sqrt{2\pi}} \frac{1}{\sigma \sqrt{2\pi}} e^{-r^2/2\sigma^2}.
\]

The terms containing \( \theta_1, \ldots, \theta_{n-1} \) may be integrated out to get

\[
p(r), \text{ but, since they are all independent of } r, \text{ they must result in}
\]
a constant $C$ that normalizes the integral to one. This constant is given by

$$C = 2 \frac{(\frac{1}{2})^{n/2}}{\sigma^n \Gamma(\frac{n}{2})},$$

and so

$$p(r) = 2 \frac{(\frac{1}{2})^{n/2}}{\sigma^n \Gamma(\frac{n}{2})} r^{n-1} e^{-r^2/2\sigma^2}, \quad 0 \leq r < \infty,$$

where $\Gamma(\cdot)$ is the Gamma function. This is the probability density function of a $\chi^2$ distribution; it is the distribution for the Euclidean length of $n$-dimensional vectors in $\mathbb{R}^n$ whose components are i.i.d. and drawn from a zero mean Gaussian random process with variance $\sigma^2$. The mean of this distribution is given by

$$\mu_r = \frac{2^{1/2}}{\Gamma(\frac{n+1}{2})} \sigma,$$

and the variance by

$$\sigma_r^2 = \left[ n - 2 \left( \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \right)^2 \right] \sigma^2.$$

Next we derive the distribution of $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, making the change of variables $u = r^2$ yields

$$p(u) = \frac{p(r(u))}{|du/dr|} = \frac{2^{1/2}}{\Gamma(\frac{n}{2})} \frac{u^{(n-1)/2} e^{-u/2\sigma^2}}{2u^{1/2}} = \frac{1}{\sigma^n \Gamma(\frac{n}{2})} u^{n/2-1} e^{-u/2\sigma^2}, \quad 0 \leq u < \infty.$$

This distribution is readily recognized as a Gamma distribution, or more particularly, a Chi-square distribution with $n$ degrees of freedom when $\sigma = 1$. For this distribution, the mean is given by $\mu_u = n\sigma^2$ and the variance by $\sigma_u^2 = 2n\sigma^4$.

Formulas (1) and (2) give the probability densities of the length and the length squared for a vector whose components are independent samples from the same Gaussian distribution with mean 0 and variance $\sigma^2$. However, what is needed for this letter is the distance between two vectors. That is, let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be independent random variables chosen from the same Gaussian distribution with mean 0 and variance $\sigma^2$, and define

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},
\quad u = r^2.$$

However, note that if $x$ and $y$ are independent Gaussian random variables with mean 0 and variance $\sigma^2$, then $(x - y)$ is also Gaussian, again with mean 0 but now with variance $2\sigma^2$. In fact, we can relax the requirement that $x$ and $y$ have zero mean. As long as they have the same mean, then the result is equally valid. Consequently, for the case of the distance between two vectors, formulas (1) and (2) become

$$p(r) = 2 \frac{(\frac{1}{2})^{n/2}}{\sigma^n \Gamma(\frac{n}{2})} r^{n-1} e^{-r^2/4\sigma^2}, \quad 0 \leq r < \infty,$$

$$p(u) = \frac{(\frac{1}{2})^{n/2}}{\sigma^n \Gamma(\frac{n}{2})} u^{n/2-1} e^{-u/4\sigma^2}, \quad 0 \leq u < \infty.$$

4. Detection strategies

Suppose we are sampling from a data stream of values of the form

$$x(m) = s(m) + \eta(m), \quad m = 1, 2, \ldots, M,$$

where the $s(m)$ are values of a signal and the $\eta(m)$ are Gaussian white noise with zero mean and variance $\sigma^2$. The signal may or may not be present, and the parameters of the signal are unknown. Two approaches will be evaluated in this Letter: the first is a straightforward normality detector and the second makes use of the recurrence matrix of the data.

4.1. Normality detector

If the signal is absent, then the measured values $x(m)$ represent just Gaussian noise $\eta(m)$. Hence the histogram generated from the $x(m)$ values should fit the Gaussian density function of the background noise. This hypothesis is tested by means of a $\chi^2$ Goodness-of-fit test based on the $\chi^2$ statistic

$$S = \sum_{k=1}^{K} \frac{(n_k - e_k)^2}{e_k},$$

where $K$ is the number of bins of the histogram and the $n_k$ and $e_k$ are the observed counts and the expected Gaussian counts in each bin, respectively. The statistic $S$ is compared to the appropriate threshold value $\chi^2_0$ of the $\chi^2$ distribution with $K - 1$ degrees of freedom. If $S < \chi^2_0$ then we declare no signal present. However, if $S > \chi^2_0$ then we declare a signal present. In the calculations to follow, $K = 15$ bins proved to be suitable for $M = 100–200$ (the size of the signals considered in this work). The threshold used in this detection scheme corresponded to a critical region of size 0.05 for the $\chi^2$ distribution.

4.2. Recurrence plot detector

In this case, the time-delay vectors $X_i$ for $i = 1, \ldots, N$ are constructed from the $x(m)$ values as described before. Each of these vectors has length $n$, where $n$ is the embedding dimension, and each component of $X_i$ is a value in the original data stream which samples are assumed to be independent. The $N \times N$ unthresholded recurrence plot has entries $d_{i,j} = ||X_i - X_j||$ which is just the distance $r$ between the vectors as described above. If there is no signal present then the original samples come from a Gaussian distribution with zero mean and variance $\sigma^2$ and so the non-diagonal entries in the unthresholded recurrence plot follow the $\chi$ distribution given in Eq. (3). To avoid duplications, we only use the $N(N - 1)/2$ entries above the main diagonal. This technique could also be based on $r^2$ values, and then Eq. (4) would be used.

The procedure is now similar to the first technique. If there is no signal present, the histogram of the $d_{i,j}$ values should fit the $\chi$ distribution of Eq. (3). Again a $\chi^2$ Goodness-of-fit test can be applied with a threshold based on the $\chi^2$ distribution with $K - 1$ degrees of freedom if $\sigma$ is known ($K - 2$ degrees of freedom if $\sigma$ is estimated from the data).

We now have many more data points than in the corresponding Normality Detector Test, but $K = 15$ bins still proved to be sufficient for a valid $\chi^2$ test. Again, if the $\chi^2$ value was below the threshold, we declared the signal to be absent; if above the threshold, we declared the signal present. In this case, however, we found that for the types of signals treated in this analysis, too many false positives occurred when comparing $S$ to the threshold $\chi^2_0$. That is, the value of $S$ was often rather large even with no signal present, but much larger when the signal was present. This effect is probably due to the redundancies in the recurrence matrix data. The derivation of the null distribution assumed independent distances when in reality these distances may contain redundancies. The result is that the size of the test (the probability that the null hypothesis is rejected) is not commensurate with the observed Type-I error. In this case, if the threshold $\chi^2_0$ is set such
that we expect 5% false alarms, the correlations in the data result in a percentage of rejections that can be much larger than 5%. We therefore calibrated our test by using an adjustment factor $\kappa$ to multiply the $\chi^2$ statistic, $S$. This factor reduces the number of false positives to the correct level thus allowing for a more meaningful test [13]. Calculations have shown that small values $n = 2, 3, 4$ of the embedding dimension work best in this detector, and for these values $\kappa = 0.5, 0.3, 0.1$, respectively. The number of data can also influence the choice of $\kappa$. We have found that larger values for $M$ require a decrease in $\kappa$ value.

Embedding dimension $n$ is one important sensitivity parameter; others are the delay time, $L$ and the sampling rate of the original data stream. Of course it is desirable that this detection technique will be effective for many signal types. The optimal values of the key sensitivity parameters will generally be a function of the type of signal received. Consequently, in practice, a bank of parameter values should be swept through to test for the presence of an unknown signal.

5. Results

Both detectors (normality and recurrence) are compared here in terms of their Receiver Operating Characteristic (ROC) performance. The ROC curve simply displays the probability of detection $P_D$ versus the probability of false alarm $P_{fa}$ associated with these detectors as a function of the detection threshold. The signal of interest was taken to be a 10 Hz sine wave with additive Gaussian noise. The signal-to-noise ratio (SNR) was taken as the average power of the signal divided by the variance of the noise.

Fig. 1 shows some typical results comparing detector performance as a function of signal to noise ratio. The embedding parameters used in this example were $M = 100$, $n = 3$, $L = 5$. The signal was sampled at intervals of $\Delta t = 0.01$ s. For the recurrence-based detector the threshold adjustment factor was set to $\kappa = 0.3$. The results of Fig. 1 are typical of those found using other embedding parameters. For low SNR values, the normality detector easily outperforms the recurrence detector. However, for intermediate SNR levels there typically exists a range where the recurrence-based detector shows superior ROC performance. Changing the embedding parameters does not seem to affect the result. For example, using a dimension of $n = 2$ and $L = 8$ and $\kappa = 0.5$ results in the ROC curves of Fig. 2.

Again, for low SNR values ($< -10$ dB) the chi-square test for non-normality is the best performer while for higher SNR values the recurrence-based detector shows the best ROC performance. The influence of the number of points and sampling rate were also explored. Fig. 3 shows the results of increasing the number of data to $M = 200$ points and decreasing the sampling interval to $\Delta t = 0.005$ s (thus the total duration of the signal is kept constant from the previous examples). The embedding parameters used were $n = 2$, $L = 8$ as in the previous example. For this number of data, an appropriate threshold value was determined to be $\kappa = 0.3$.

As before there exists a range of SNR values below which the normality detector yields the more powerful result. Above $-12$ dB
Fig. 2. ROC curves associated with detecting a 10 Hz. sine-wave in Gaussian noise for varying SNR levels. Sampling parameters were \( M = 100 \) and \( \Delta t = 0.01 \) s. Embedding parameters were \( n = 2, L = 8 \). Threshold factor was \( \kappa = 0.5 \).

Fig. 3. ROC curves associated with detecting a 10 Hz. sine-wave in Gaussian noise for varying SNR levels. Sampling parameters were \( M = 200 \) and \( \Delta t = 0.005 \) s. Embedding parameters were \( n = 2, L = 8 \). Threshold factor was \( \kappa = 0.3 \).
Fig. 4. ROC curves associated with detecting a 10 Hz. Square-wave in Gaussian noise for varying SNR levels. Sampling parameters were $M = 200$ and $\Delta t = 0.005$ s. Embedding parameters were $n = 2$, $L = 8$. Threshold factor was $\kappa = 0.3$.

Fig. 5. ROC curves associated with detecting the output of the Lorenz system in Gaussian noise for varying SNR levels. Sampling parameters were $M = 200$ and $\Delta t = 0.005$ s. Embedding parameters were $n = 2$, $L = 8$. Threshold factor was $\kappa = 0.3$. 
SNR, however, the recurrence-based detector again shows superior performance until both methods converge to unity for the probability of detection for any probability of false alarm.

The results shown above hold for a variety of both embedding and sampling parameters. For \( n > 3 \), however, the redundancies in the recurrence plot become much larger so that the threshold adjustment factor must be set to values \( k < 0.1 \). It is therefore apparent that there does exist a range of SNR values for which the recurrence-based detection schemes have some merit over the more general normality detector in detecting the presence of a sine-wave in Gaussian noise. It should be stressed, however, that the value of these detectors is that they do not assume a priori knowledge of the incoming signal. If it was known that a sine-wave was the signal of interest an optimal detector can be derived, depending on how much is known about the sine-wave (frequency, phase, etc.).

As another test of the recurrence-based detector, the detection of a square wave in additive Gaussian noise was considered. The embedding parameters \( n = 2 \), and \( l = 8 \) were used along with the sampling parameters of \( M = 200 \), \( \Delta t = 0.005 \). Fig. 4 shows these results. Even for a very different type of periodic signal, the same trend in ROC curves persists as a function of SNR. Finally, the detection of a deterministic, aperiodic signal was considered. Specifically, we chose the signal to be an \( M = 200 \) point sample of the first state variable \( x_1(t) \) of the chaotic Lorenz system of equations:

\[
\begin{align*}
\dot{x}_1 &= 48(x_2 - x_1), \\
\dot{x}_2 &= 120x_1 - 3x_2 - 3x_1x_3, \\
\dot{x}_3 &= -12x_3 + 3x_1x_2.
\end{align*}
\]

The parameters of this system equations were chosen such that the signal is operating in the chaotic regime with a frequency content commensurate with the \( \Delta t = 0.005 \) sampling interval. Again, one sees (Fig. 5) the same general performance of the recurrence-based detector as compared to the more traditional approach. For very low SNR values the recurrence-based approach has difficulty, however as the SNR value increases there is clear improvement in the probability of detection for low false alarm probabilities.

6. Summary

This work has focused on the detection of signals in Gaussian noise using a detector based on the signal's recurrence matrix. Under the null hypothesis of no signal, the probability distribution for the entries of the recurrence matrix was derived and found to be a \( \chi^2 \) distribution. Using this distribution, a chi-squared test was performed in order to assess whether or not some underlying signal was present. Due to redundancies in the recurrence plot, this test had to be slightly modified in order to produce a meaningful Type-I error. Receiver Operating Characteristic (ROC) curves were then used to assess the performance of this detector as a function of embedding parameters, signal sampling parameters, and signal-to-noise ratio (SNR). Comparisons were made between the recurrence-based detector and a simple normality detector (based on the chi-squared test) in terms of their ability to evince non-normality in the incoming signal. Results indicated that there is a range of SNR values for which the recurrence-based detector offers superior performance (in the sense of lower Type-II error for a given Type-I error) over the normality detector. This behavior was observed for both sine wave and square wave signals as well as for an aperiodic, chaotic signal buried in additive Gaussian noise.

References