Use of Recurrence Plots in the Analysis of Time-Series Data

Matthew Koebbe  
Department of Mathematics  
University of California - Santa Cruz  
Santa Cruz, CA 95064

and

Gottfried Mayer-Kress  
Department of Mathematics  
University of California - Santa Cruz  
Santa Cruz, CA 95064

and

Joe Zbilut  
Physiology Department  
Rush - Pts. - St. Luke’s Medical Center  
Chicago, IL 60612

September 29, 1994
Abstract

Recovering information about the periodic behavior of a non-stationary signal typically requires a large data set and extensive off-line analysis. Alternatively, we’ve modified Eckmann, Kamphorst and Ruelle’s idea of a recurrence plot, which they defined to appreciate the dynamics of a time series without either of these constraints. Geometrical properties of our plots provide dimensional complexity and local rate of divergence estimates.
1 Introduction

Experimentalists dealing with the nonlinear dynamics of biological data have been at once encouraged and disappointed with developments over the past ten years: whereas parameters such as Lyapunov exponents, and dimensions have suggested new insights into biological processes, results have also been equivocal due to the assumptions that the data series are autonomous and that their lengths are much longer than the characteristic times of the system in question. [1] [2] By and large, such assumptions cannot be supported.

A new graphical tool for the representation of erratic time series was introduced in 1987 by Eckmann, Kamphorst, and Ruelle [3] Here they begin with a time-ordered sequence of vectors in $\mathbb{R}^n$, and let $x(i)$ denote the $i$-th point on the orbit. They define a two-dimensional array over $\mathbb{Z}_2$. A value of "1" is assigned to the $(i,j)$-th position in the array if and only if $x(j)$ is sufficiently close to $x(i)$, i.e. if and only if $x(j)$ falls within a ball of radius $r(i)$ centered at $x(i)$, where $r(i)$ is chosen by some suitable method (For example, choose the radius such that 10 other data points lie within the ball.) This array of points is then known as a "recurrence plot". Their recurrence plots tend to be nearly symmetric with respect to the diagonal $i = j$ because, in general, if $x(i)$ is close to $x(j)$ then $x(j)$ is usually "close" to $x(i)$. (Notice that the plot is not necessarily symmetric as $r(i)$ does not have to equal $r(j)$.) Large-scale diagonal "lines" parallel to $i = j$ indicate periodic behavior which is not always visible in the original data. At the same time, the length of small-scale line segments parallel to $i = j$ were claimed to be proportional to the inverse of the largest positive Lyapunov exponent. Additionally, drift in the signal is recognized by the over-all reduction of recurrences away from the diagonal.

Eckmann et. al. mentioned in their paper, that the vectors $x(i)$ could be reconstructed from a time-series using the method of time delays. Since we intend to limit ourselves to time-series data, it is useful to briefly recall this method here. We suppose our data consist of a time-ordered sequence of integer values, $d(i)$. In order to gain insight as to how such a data sequence could arise, we suppose that it has been generated by some dynamic in $\mathbb{R}^n$. In other words, we suppose that there are $n$ independent variables evolving in time according to some deterministic equation. The time series can be seen as a projection of these variables to a one-dimensional observable. We reconstruct the $n$-dimensional dynamics using the now common time-delay embedding method of Packard, et. al. [4]. In general, $n$-dimensional vectors can be reconstructed from a time-series $\{d(i)\}_{i=1}^{N_d}$, with a delay of $\tau$, as $\tilde{v}_i = (d(i), d(i+\tau), d(i+2\tau), \ldots, d(i + n\tau - 1))$ (where $N_d$ is the number of data points originally provided.) It is known that a value for $\tau$ which coincides with the first minimum of the mutual information content provides a good estimate for a reliable reconstruction [5]. (For a thorough explanation of the relationship between the delay, $\tau$, and the amount of new information provided by each subsequent vector, see R. Shaw's, The Dripping Faucet as a Model Chaotic System.
We have modified this algorithm by neglecting the assymetry and fixing the size of the neighborhoods. Note that since this produces a symmetric array, it suffices to refer to the upper left triangle. In order to display as much information as possible, we create our recurrence plot by letting the i-th column represent the distance from $\overline{r}_{i-1}$ to the next $h$ successive vectors forward in time, where $h$ is the height of the array. We assign a value of "1" to those distances which are less than or equal to a constant distance. Note that our plots can be recovered from the plots of Eckmann, et. al. by rotating the vertical axis 45 degrees (and by fixing the size of the neighborhoods.)

We have written a C language program which runs on a Personal Iris allowing us to choose an embedding dimension, a delay, a particular norm, and the number of vectors displayed. [6] Once the plot has been drawn, we can select a particular array element within an interesting geometrical feature. This opens an interpretive window below which displays a neighborhood of the data points used to reconstruct the two indicated vectors (super-imposed in different colors). The data points actually used to re-construct the chosen vectors are highlighted in still different colors. Thus, in the case where the time series has been generated by sampling one continuous signal, we are able to directly compare geometric features appearing in the recurrence plot with the portion(s) of the signal which generated them.

One of our earlier observations was the influence of choosing different norms when creating these plots. In order to understand the effect that choosing a different norm has on a recurrence plot, let us consider the effect of three standard norms: the $l_1$, the Euclidean ($l_2$), and the supremum norm ($l_{sup}$). Recall that, in two dimensions, the unit ball as measured by each of these norms is a parallelogram with vertices at $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, a disk of radius one $(1)$ centered at the origin, and a square whose edges lie on $x = \pm 1$, $y = \pm 1$, respectively. Thus, the supremum norm is most likely to determine that a vector in the reconstruction is within a $\delta$-neighborhood of another vector. This norm serves to emphasize the outline of the reconstruction, i.e. edge-effects predominate. Alternatively, for a fixed $\delta$, the $l_1$ norm should find the fewest number of recurrences in a given reconstruction. Hence, using this norm reveals more information about the local behavior of the reconstruction. The Euclidean norm has intermediate effects. These considerations are of some concern since embedding a time series in higher dimensions causes the distances to be distributed over a narrow range near the surface of the sphere. Thus because of the relative rarity of recurrences when using the $l_1$ norm, routinely embedding a time series in $\mathbb{R}^n$, where $n > 3$, causes an increase in observed recurrences to becomes more significant.
2 Recurrence Plot Structure Classification

We've primarily used the $l_1$ norm to determine the distance between vectors, so we let $l_{i,j}$ denote the distance from $\vec{v}_i$ to $\vec{v}_j$. We assume that $(x, y) = (0, 0)$ is in the lower left-hand corner. Notice that if the point $(x, y)$ on our plot corresponds to the distance $l_{i,j}$, then $(x + 1, y)$ corresponds to $l_{i+1,j+1}$, by definition. By default, we normalize the data set so that it corresponds to 8 bits of information, and assign a value of 1 to those distances which correspond to an information gain of less than 4 bits.

One of the more obvious, and most important, features in our plots are horizontal line segments. Such line segments correspond to sequences of points $(x, y), (x + 1, y), \ldots, (x + k, y)$ which, in turn, correspond to sequences of distances $l_{i,j}, l_{i+1,j+1}, \ldots, l_{i+k,j+k}$. If we suppose that, at least locally, the system is described by a hyperbolic dynamic, then we know that these vectors are diverging at the rate of $e^\lambda$, where $\lambda$ is the largest positive local "Lyapunov exponent". Thus $k$, the length of this line segment, defines an upper bound on this exponent:

$$\lambda < C \cdot T \cdot \frac{1}{k+1},$$

where: $C$ (bits) is the user-defined cutoff, $T$ (cycles/sec.) is the sampling rate, and $k$ (cycles) is the number of array elements in the horizontal line segment. Shaw has pointed out [7] that positive Lyapunov exponents imply noise amplification, and the production of new macroscopic information through the amplification of small fluctuations. The number of calculations needed to determine such an estimate by other methods is typically much greater. We also note that this feature is readily discernible in data sets containing as few as 400 points. Note that the worst-case scenario involved in calculating this bound assumes that the two vector sequences come arbitrarily close to each other. On the other hand, realizing how near to parallel these trajectories can be demonstrates why no lower bound, other than zero, is possible.

An often-noticed feature in lower-dimensional embeddings of complex signals, is a number of line segments: both vertical and diagonal proceeding from the upper left to lower right. These segments correspond to sequences of points $(x, y), (x + 1, y - 1), \ldots, (x + k, y - k)$ which, in turn, correspond to sequences of distances $l_{i,j}, l_{i+1,j+1}, \ldots, l_{i+k,j+k}$. Thus, such a line indicates that $\vec{v}_i$ is close to several successive vectors. Similarly, a vertical line tells us that $\vec{v}_i$ is close to several successive vectors. Upon using the interpretive window, we notice that in low-dimensional embeddings of certain signals, the vectors which remain close to several successive vectors come from data which are near the over-all average value, while vectors which appear to be far from several successive others consist of data points taken from the extremes of the data set. This indicates that our technique acts as a threshold detector in these lower dimensions. This phenomena disappears as we approach the "proper" embedding dimension. In addition to seeing this effect in tests with data generated by the x-coordinate of
the Henon, the Lorenz, and other well-known systems, an excellent example is provided by a particular set of EEG data [2]. In \( \mathbb{R}^{15} \) with a delay of four (4), we see broad horizontal bands of dots alternating with regions of emptiness indicating a high-dimensional periodicity in the data. In contrast, in a 1-dimensional embedding with a delay of 1, we see numerous horizontal and upper left to lower right diagonals.

Another interesting, yet relatively rare, pattern to observe is the diagonal line segments proceeding from the lower-left to upper-right. This pattern was originally noticed in generated data which produced solid diamond-shaped regions in our plots. These regions can be viewed in two distinct ways: viewed as successive line segments which have a negative slope, these diamond-shaped regions represent successive vectors which remain close to a sequence of successive vectors elsewhere in the signal. However, these regions can also be viewed as a sequence of line segments with positive slope. Where these segments are actually have a slope of 1, they correspond to sequences of points: \((x, y), (x + 1, y + 1), \ldots, (x + k, y + k)\), which, in turn, correspond to sequences of distances: \(l_{(i,j)}, l_{(i+1,j+1)}, \ldots, l_{(i+k,j+k)}\). Thus, these segments indicate harmonic oscillations within the data, i.e. some portion of the original signal has frequency \(1/T\) and another portion has frequency \(n/T\). Equivalently, one can vary the sampling rate of a sin wave and experimentally reproduce this phenomenon. If the sampling rate is made to oscillate each period between a constant value and double this value, then diagonal lines as described above arise in the recurrence plots. As the ratio of the two periods is varied, line segments with different positive slopes can be made to appear.

In addition to local rate of divergence information provided by the lengths of horizontal line segments, a recurrence plot also provides an estimate of the local dimension of the signal in a more precise sense. Suppose that instead of defining our array over \( Z_2 \), we calculate and order all the distances between pairs of reconstructed vectors derived from a particular data set. We then have a natural correspondence between the informational difference between two reconstructed vectors and \( Z_{256} \). The elements of this field, then, can be made to correspond to a map of 256 colors which allows us to produce colored recurrence plots. (Note: In effect, this allows us to visualize another system variable.) Consider the \( i \)-th column of a particular colored recurrence plot. Notice, that every appearance of a particular color, within that column, corresponds to another vector, out of the next \( h \), which comes within a neighborhood (or “ball of radius \( r(i-1) \)) of vector \( \vec{v}_{i-1} \). Thus each column of a colored recurrence plot acts as an approximate gauge function. Recently, Thieker showed that although the over-all error resulting from typical dimension estimations scales as \( \frac{1}{\sqrt{N}} \), where \( N \) is the number of data points, this error typically scales as \( \frac{1}{N} \) for small \( n \). [8]. Based on these ideas, we are currently developing an algorithm to track the local gauge function variation.
3 Conclusion

Constructing a recurrence plot in the fashion we’ve described is an efficient means of recovering dynamical information from a reconstructed orbit. This method does not require as much data as more traditional methods, and is computationally quick, as well. Qualitative information about local divergence rates and relative dimensionality of the original signal are visually obvious. Furthermore, this method emphasizes that adiabatic systems may be usefully analyzed without requiring the reconstruction of an entire attractor.

References