Multiwavelet scale multidimensional recurrence quantification analysis

Cite as: Chaos 30, 123109 (2020); doi: 10.1063/5.0025882
Submitted: 19 August 2020 · Accepted: 10 November 2020 ·
Published Online: 2 December 2020

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ABSTRACT
The multiwavelet scale multidimensional recurrence quantification analysis (MWMRQA) method is proposed in this paper, which is a combination of multidimensional recurrence quantification analysis and wavelet packet decomposition. It allows us to quantify the recurrence properties of a single multidimensional time series under different wavelet scales. We apply the MWMRQA method to the Lorenz system and the Chinese stock market, respectively, and show the feasibility of this method as well as the dynamic variation of the Lorenz system and the Chinese stock market under different wavelet scales. This provides another perspective for other disciplines that need to study the recurrence properties of different scales in the future.

In this paper, we introduced the multiwavelet scale multidimensional recurrence quantification analysis (MWMRQA) method. We applied it to the model system and the Chinese stock market. These experiments show the feasibility of the method. Our method can provide another perspective for researchers who want to study time series with higher complexity or deterministic, that is, a certain time series has higher deterministic or complexity at a certain wavelet scale and then let them use the time series under this wavelet scale to carry out their own research.

I. INTRODUCTION
As we know, recurrence is the fundamental characteristic of many dynamic systems. It has been studied and employed for a long time in the field of chaos theory. The use of powerful computers has promoted the development of chaos theory, which allows this digital tool to deal with tedious calculations that require the concept of recurrence for more practical purposes. With the intense usage of these digital tools, the similarity matrix was reinvented by several scientific disciplines around the change from the 1970s to 1980s. As a result, in 1987, Eckmann et al. used the similarity matrix as a tool to visualize the recurrence of the high-dimensional phase space trajectory, which established a new direction in nonlinear data analysis, namely, recurrence plot (RP). The structure of recurrence points, diagonal, and vertical horizontal lines in the recurrence plot can show the characteristics of the system dynamics. In order to further understand the changing process of the system dynamics, the recurrence quantification analysis (RQA) method is proposed, which quantifies the structure in the RP and can have a deeper understanding of the dynamic state variation of the system.

RP and RQA methods are used to study the autocorrelation of one-dimensional time series. In order to study the cross-correlation between two one-dimensional time series, the cross recurrence plot (CRP) and cross recurrence quantification analysis (CRQA) methods are proposed. They are binary extensions of RP and RQA, respectively, and are used to describe the change process of the system dynamic state between two one-dimensional time series.

However, RP, RQA, CRP, and CRQA methods are all studied on one-dimensional time series. It is obvious that these methods are not suitable for studying the dynamic behavior of multiple variables. To overcome this problem, in 2016, Wallot et al. proposed a multidimensional recurrence quantification analysis (MDRQA) method, which is used to describe the behavior of time series generated by multiple interdependent variables, potentially exhibiting nonlinear behavior over time.

The time series produced by the complex system in the real world (such as financial time series, EEG, and traffic signals) are often have high fluctuation and unpredictable with strong nonlinearity and nonstationary. In order to resolve the high volatility of these time series, the wavelet packet decomposition (WPD) has caught the eyes of scholars. WPD is a further extension of...
discrete wavelet transform (DWT). It not only divides the approximate space but also the detailed space at each level. This provides better resolution on both time and frequency scales. With respect to denoising, WPD can decompose nonstationary, nonlinear, and complex sequences into a set of low-frequency subseries. In recent years, some studies have pointed out some multi-scale nonlinear features hidden in financial markets, in response to this practical problem and the wide application of wavelet analysis methods in the economic and financial fields. This paper proposes a method to investigate the dynamic state changes of multidimensional systems under different wavelet scales—multiwavelet scale multidimensional recurrence quantification analysis (MWMRQA) based on WPD. We apply the MWMRQA method to the Chinese stock market to study the dynamic changes of the Chinese stock under different wavelet scales. We also applied MWMRQA to the model system to verify the feasibility of this method.

The remainder of the paper is organized as follows: in Sec. II, we introduce the MWMRQA method. The empirical results and analysis are presented in Sec. III. Finally, Sec. IV offers concluding remarks.

II. METHODOLOGY
A. Wavelet packet decomposition

WPD is a time–frequency domain analysis method, which has advantages in the process of analyzing nonlinear nonstationary signals. It is an extended method of DWT, which can decompose long-term time series into several subseries at different wavelet scales. The nonlinear calculations performed under these multiwavelet scales are plausible and effective. The wavelet function is shown as follows:

\[
\begin{align*}
\mu_0(t) &= \sqrt{2} \sum_{k \in \mathbb{Z}} h(k) \mu_0(2t - k), \\
\mu_1(t) &= \sqrt{2} \sum_{k \in \mathbb{Z}} g(k) \mu_0(2t - k),
\end{align*}
\]

where \( h(k) \) is a low-pass filtering function, \( g(k) = (-1)^k h(1 - k) \) is a high-pass filtering function. The wavelet basis function and scaling function are denoted as \( \phi(t) \) and \( \psi(t) \), respectively, and \( \phi(t) = \mu_0(t) \) and \( \psi(t) = \mu_1(t) \).

The wavelet function satisfies the following double scale function:

\[
\begin{align*}
\mu_{2n}(t) &= \sqrt{2} \sum_{k \in \mathbb{Z}} h(k) \mu_n(2t - k), \\
\mu_{2n+1}(t) &= \sqrt{2} \sum_{k \in \mathbb{Z}} g(k) \mu_n(2t - k).
\end{align*}
\]

The decomposition algorithm is defined as follows:

\[
\begin{align*}
d_{2n}^{j+1}(k) &= \sum_{t \in \mathbb{Z}} h(t - 2k) d_{2n}^j(t), \\
d_{2n+1}^{j+1}(k) &= \sum_{t \in \mathbb{Z}} g(t - 2k) d_{2n}^j(t),
\end{align*}
\]

where \( d_{2n}^j \) and \( d_{2n+1}^j \) are the two decomposition coefficients.

The selection of wavelet basis function is very important in the wavelet analysis of signals. Choosing different wavelet basis functions will produce different denoising effects. Our commonly used wavelet basis functions include Haar wavelets, Daubechies (DB) series wavelets, etc. Among them, the DB series wavelets are compactly supported and orthogonal wavelet bases, which are very suitable for processing nonstationary signals. The article points out that using Daubechies (DB2) basis functions to decompose stock market indexes can better generate less noisy sequences for a given financial dataset, as shown in Ref. 27. Usually, DB series wavelets are used to do wavelet packet decomposition of the Lorenz system. Therefore, in the following two experiments, we will select the DB2 basis function for the level-3 wavelet packet decomposition.

The level-3 wavelet packet decomposition process of signal \( S \) is presented in Fig. 1, where “A” represents the low-frequency component, “D” represents the high-frequency component, and the subscript number represents the number of level of the wavelet packet decomposition.
As shown in Fig. 1, S is the original signal. Then, the level-1 WPD is to decompose the signal S into the low-frequency component on the $A_1$ wavelet scale and the high-frequency component on the $D_1$ wavelet scale. The level-2 WPD is carried out on the basis of the level-1 WPD, that is, the high-frequency component $A_1$ and the low-frequency component $D_1$ of the level-1 are decomposed into the respective high-frequency and low-frequency components. As shown in this figure, the wavelet scales $AA_2$ and $AD_2$ are the low-frequency and high-frequency components decomposed from the level-1 low-frequency component $A_1$, respectively; the wavelet scales $DA_2$ and $DD_2$ are the low-frequency and high-frequency components decomposed from the level-1 high-frequency component $D_1$, respectively. By analogy, we can realize the whole process of wavelet packet decomposition.

**B. Multidimensional recurrence quantification analysis**

MDRQA is a multivariable extension of simple RQA. It is a technique that can analyze the nonlinear behavior of time series composed of multiple variables. Given a d-dimensional ($d > 1$) time series $X$,

$$X = \begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
\vdots \\
X_m
\end{pmatrix} = \begin{pmatrix}
X_{1,1} & X_{1,2} & X_{1,3} & \ldots & X_{1,d} \\
X_{2,1} & X_{2,2} & X_{2,3} & \ldots & X_{2,d} \\
X_{3,1} & X_{3,2} & X_{3,3} & \ldots & X_{3,d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_{m,1} & X_{m,2} & X_{m,3} & \ldots & X_{m,d}
\end{pmatrix}, \quad (4)$$

where each column vector in $X$ is a separate time series and Eq. (4) represents unembedded time series. If we select the appropriate embedding dimension $D$ and time delay $\tau$, we can also investigate the embedded time series. The embedded phase space portraits for

![FIG. 2. The Lorenz system: $\sigma = 10$, $\beta = \frac{8}{3}$, and $\mu = 28$.](image)

![FIG. 3. MWMRQA measures of the Lorenz system: (a) RR, (b) DET, (c) ENTR, and (d) LAM, $\tau = 1, D = 3$, and $r = 1$.](image)
multidimensional time series $X$ is as follows:

$$V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix} = \begin{pmatrix} X_1 & X_{1+\tau} & X_{1+2\tau} & \cdots & X_{1+(D-1)\tau} \\ X_2 & X_{2+\tau} & X_{2+2\tau} & \cdots & X_{2+(D-1)\tau} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_m & X_{m+\tau} & X_{m+2\tau} & \cdots & X_{m+(D-1)\tau} \end{pmatrix}, \quad (5)$$

where $\tau$ is the time delay and $D$ is the embedding dimension, and they are evaluated by the extended average mutual information (AMI) and false-nearest neighbor (FNN) functions, respectively.

The multidimensional recurrences (MDRs) of a multidimensional recurrence plot (MDRP) can be defined as the comparison of the distances between all possible pairs of values for the time series $X$ (or reconstruct phase space portraits $V$) with the threshold $r$. Threshold $r$ is a crucial parameter for MDRP. If $r$ is chosen too small, there may be almost no recurrence points in MDRP that we cannot learn anything about the recurrence structure of the underlying system. On the other hand, if $r$ is chosen too large, almost every point is a neighbor of another point, which leads to a lot of artifacts. Therefore, special attention has to be given for its selection. In the phase space, if $dists \leq r$, the state is recurrent. Otherwise, the state is not recurrent. The MDR is calculated from the following formula:

$$MDR_{ij}(r) = \Theta(r - ||X_i - X_j||), \quad (6)$$

where $j = 1, \ldots, m$, $\Theta$ is the Heaviside function, $r$ is the arbitrary threshold distance, $|| \cdot ||$ is a norm (like $L_1$-norm, Euclidean norm, and maximum or supremum norm). In this paper, we choose the Euclidean norm. The MDRP can be drawn by Eq. (6). In order to further quantify recurrence structures such as recurrence points, diagonal, vertical, and horizontal lines in MDRP, the following four recurrence measurements are used to quantify them:

1. **Recurrence Rate (RR)**
   - The recurrence rate quantifies the percentage of recurrence points within a specified radius. The variable ranges from 0% (no recurrence point) to 100% (all recurrence points). From its changes, we can see the probability of similar state occurrence with a certain delay in the system. The higher the value of RR, the higher the probability of the same state in its process.

2. **Determinism (DET)**
   - DET measures the ratio of recurrence points that form a diagonal structure. The periodic signal produces a long diagonal, the chaotic signal produces a very short diagonal and the random signal does not produce a diagonal at all.

3. **Entropy (ENTR)**
   - ENTR is Shannon information entropy of all diagonal lengths distributed on integer binary bits in histogram. ENTR is a measure of signal complexity and is calibrated in bits/bin. For each non-zero box, the probability of a single histogram box is calculated, and then the sum is obtained according to Shannon’s equation.

4. **Laminarity (LAM)**
   - The LAM measures the percentage of recurrence points consisting of vertical line structures. The line parameter still has to govern the minimum length of the included vertical line.
C. Multiwavelet scale multidimensional recurrence quantification analysis

In order to uncover dynamics of different group levels under multiple wavelet scales, we introduced the MWMRQA method, which combines WPD with MDRQA. The MWMRQA process is as follows:

1. Given a multidimensional time series $X$ [Eq. (4)], first, we decompose each column of independent time series in $X$ by wavelet packet decomposition. Then, we get several subseries with different wavelet scales. In this paper, we carry out the level-3 wavelet packet decomposition and get eight different wavelet scales. Then, the column vector time series on each wavelet scale are combined according to the column vector sequence in the original $X$ sequence to form eight pairs of multidimensional time series $X_m$ ($m = 1, \ldots, 8$) with different wavelet scales.

2. Apply the MDRQA process described above to these eight pairs of multidimensional time series $X_m$ ($m = 1, \ldots, 8$) at different wavelet scales and then we also can get four MWMRQA measurements under different wavelet scales. The four MWMRQA measurements include: RR, which quantifies the percentage of recurrence points within a specified radius at different wavelet scales; DET, which measures the ratio of recurrence points that

![Graphs](a) (b) (c) (d)

**FIG. 5.** The (a) daily closing price, (b) daily opening price, (c) daily lowest price, and (d) daily highest price of SZSE.
form a diagonal structure at different wavelet scales; ENTR, which is Shannon information entropy of all diagonal lengths distributed on integer binary bits in histogram under different wavelet scales; and LAM, which measures the percentage of recurrence points consisting of vertical line structures at different wavelet scales.

III. RESULTS AND ANALYSIS

A. Applying MWMRQA to the Lorenz system

The process of the MWMRQA method has been described above. Now, we want to show how to use the MWMRQA method to infer the multidimensional dynamics of the system at different wavelet scales by considering multiple observables. As an example, we choose the Lorenz system, a dynamic system composed of three coupled differential equations as follows:

\[
\begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= \mu x - y - xz, \\
\dot{z} &= xy - \beta z.
\end{align*}
\]

Here, we set the same parameters as Wallot, \(\sigma = 10, \beta = \frac{8}{3}, \mu = 28\), which represents that the Lorenz system is chaotic. The resulting dynamic of the Lorenz system is presented in Fig. 2. In the article, \(\tau = 1, D = 3, \text{ and } r = 1\). Therefore, we set the same parameters in our paper.

Figure 3 shows the four MWMRQA measurements at different wavelet scales. We record the symbols of the level-3 wavelet packet decomposition in Fig. 1 as the names of different wavelet scales in our experiment. For RR measurement in Fig. 3(a), the values of RR at \(\text{AAA}_3, \text{AAD}_3, \text{ADA}_3, \text{DDD}_3\) scales are equal and all have the highest value, which indicates at these wavelet scales, the probability that a state recurs to its \(r\)-neighborhood in Lorenz phase space is the same and the largest. At the \(\text{DAA}_3\) scale, the RR value is lowest, which indicates the probability that a state recurs to its \(r\)-neighborhood in the Lorenz phase space is the minimum at this wavelet scale. The DET, ENTR, LAM curves approximate the shape of a letter W. From the \(\text{AAD}_3\) scale to the \(\text{DAA}_3\) scale, the DET curve roughly shows a downward trend, which indicates that in this wavelet scale range, the determinism and predictability of the Lorenz system are decreasing. At \(\text{AAA}_3, \text{AAD}_3, \text{DDD}_3\) scales and the \(\text{ADA}_3\) scale, the DET values reach maximum and minimum, respectively, which indicates the determinism and predictability of the Lorenz system reach highest at \(\text{AAA}_3, \text{AAD}_3, \text{DDD}_3\) scales and the lowest at the \(\text{ADA}_3\) scale. Similarly, at \(\text{AAA}_3, \text{AAD}_3, \text{DDD}_3\) scales and the \(\text{ADA}_3\) scale, the ENTR values reach maximum and minimum, respectively, which indicates the complexity of the Lorenz system reach highest at \(\text{AAA}_3, \text{AAD}_3, \text{DDD}_3\) scales and the lowest at the \(\text{ADA}_3\) scale. For the LAM measurement, the LAM values are maximum under
AAA, AAD, DDD scales and the minimum under the ADD scale, which indicates the time for slow change of the Lorenz system is longest at AAA, AAD, DDD scales and the shortest at the ADD scale.

Next, we will take another perspective to explore the dynamic state variation of the multidimensional Lorenz system at different wavelet scales. We investigate the scale effects by adjusting the radius parameters $r$; Fig. 4 shows the four MWMRQA measures vs threshold $r$ at different wavelet scales of the Lorenz system. As the threshold increases, the curves of RR, DET, ENTR and LAM measurements under different wavelet scales all show an upward trend. Among them, with the increase of the threshold, the trends of RR of AAA, AAD, AAD, DDD scales completely coincide. This shows that the probability trend of state recurrent to $r$ neighborhood in the Lorenz phase space is the same with the increase of threshold. With the increase of threshold value, the changing trend of RR on DAA, DAD, DDA wavelet scales is similar, but the changing trend of RR in the ADD scale is changeable. When the threshold value of the ADD scale increases to 0.3, the changing trend of RR is similar to that of DAA, DAD, DDA wavelet scales. With the increase of threshold, the trend of DET is similar in AAA, AAD, DAD, DDA scales and ADA, DAA scales, respectively, but it is variable in the ADD scale. This shows that the changing trend of the determinism and predictability of the Lorenz system is similar under AAA, AAD, DAD, DDA scales and ADA, DAA scales, respectively, but it is variable in the ADD scale. For the ENTR measurement, when the threshold value increases to 0.5, the trend of ENTR curves under all wavelet scales tends to be stable, which shows that when the threshold value increases to 0.5, the complexity of the Lorenz system changes under all wavelet scales are similar and stable. When the threshold value increases to 0.6, the LAM curves under all wavelet scales gradually become stable and coincident, which indicates that the time for slow change of the Lorenz system tends to be stable and the same at all wavelet scales when the threshold value is increased to 0.6.

B. Applying MWMRQA to the Chinese stock market

In this section, we apply MWMRQA to the Chinese stock market to investigate the dynamic variation of financial time series under different wavelet scales. We select the daily opening price, daily highest price, daily lowest price, and daily closing price of Shenzhen component Index (SZSE). The data come from Shenzhen component Index (SZSE) website (http://www.10jqka.com.cn/) and covers the data for the 28 years from April 3, 1991 to November 1, 2019, for a total of 7013 days. Figure 5 shows the time series of daily opening price, daily highest price, daily lowest price, and daily closing price of the SZSE. The literature points out that it is appropriate to set time delay $\tau$ as 1 for financial time series and $D = 4$ by the extended AMI method. Therefore, we set $\tau = 1$, $D = 4$, and $r = 0.1$.

Figure 6 depicts the MWMRQA measurements of SZSE under different wavelet scales. The RR value of the AAA scale is the largest, and it is the smallest and equal in DAA, DAD scales, which indicates that the probability of the state returning to the $r$ neighborhood of SZSE phase space is the largest in the AAA scale and the smallest in the DAA and DAD scales. The DET represents determinism and predictability of SZSE. From the AAA scale to the ADA scale, the determinism and predictability of SZSE are getting lower, and it is the same from the DAA scale to the DAD scale. From the ADA scale to the DAA scale, the determinism and predictability of SZSE are increasing, and it is the same from the DAD scale to the DDD scale.
The determinism and predictability at $AAA_3$ are highest, at $ADD_3$ is the lowest. Complexity of SZSE is represented by ENTR measure. From the $AAA_3$ scale to the $ADD_3$ scale, the complexity of SZSE is decreasing. From the $ADD_3$ scale to the $DDD_3$ scale, the complexity of SZSE is gradually increasing. LAM represents the time for slow change of SZSE. From the $AAA_3$ scale to the $ADD_3$ scale, this kind of time for SZSE is getting shorter, and it is the same from the $DAAD_3$ scale to the $DDD_3$ scale. For the $ADD_3$ scale to the $AAA_3$ scale, this kind of time for SZSE is growing.

For different thresholds $r$, the four recurrence indices of SZSE show different information. Figure 7 shows under different wavelet scales the statistical behavior of RR, DET, ENTR, and LAM measurements. Calculated from a threshold of 0.01–0.1. We observed that as the threshold increases, the RR, DET, ENTR and LAM measures of SZSE are almost the largest at the $AAA_3$ scale, and its curves are also the smoothest. For the RR measures in other wavelet scales, their trends are similar. Except for the $AAA_3$ scale, the DET, ENTR, and LAM curves under other wavelet scales all begin to fluctuate in a small range or even stabilize when the threshold is increased to 0.05. This shows that when the $r > 0.05$, determinism, predictability, complexity, and the time for slow change of SZSE are all undergoing smooth changes.

IV. CONCLUSION

The multiwavelet scale multidimensional recurrence quantification analysis (MWMRQA) method is introduced in this article, which can study the variation of the multidimensional system dynamics in different wavelet scales. We applied the MWMRQA method to the Lorenz system and the Chinese stock market, respectively. From the Lorenz experiment, we found that the probability of a state recur to its $r$ neighborhood, the determinism, predictability, and complexity; the time for slow change of the Lorenz system is largest at $AAA_3$, $ADD_3$, $DDD_3$ scales, whereas the determinism, predictability, and complexity of Lorenz are smallest at the $ADA_3$ scale, the probability of a state recur to its $r$ neighborhood, and the time for slow change of Lorenz is smallest and shortest at $DAAD_3$, $ADD_3$ scales, respectively. With the increase of the threshold value, the changing trend of RR on $DAAD_3$, $DAD_3$, $AAA_3$ wavelet scales is similar, but in the $ADD_3$ scale, it is changeable. The changing trend of the determinism and predictability of the Lorenz system is similar under $AAA_3$, $AAD_3$, $ADD_3$, $AAA_3$ scales and $ADA_3$, $DAAD_3$ scales, respectively, but it is variable in the $ADD_3$ scale. When the threshold value increases to 0.5, the complexity of the Lorenz system changes under all wavelet scales are similar and stable. The time for slow change of the Lorenz system tends to be stable and the same at all wavelet scales when the threshold value is increased to 0.6.

From the experiment of the Chinese stock market, we found that the probability of the state returning to the $r$ neighborhood, the determinism, and the predictability of SZSE is the largest in the $AAA_3$ scale. From the $AAA_3$ scale to the $ADD_3$ scale, the complexity of SZSE is decreasing, and the time for slow change of SZSE is getting shorter. By adjusting the threshold, we found that the RR, DET, ENTR, and LAM measures of SZSE are almost the smoothest at the $AAA_3$ scale. Except for the $AAA_3$ scale, when the $r > 0.05$, determinism, predictability, complexity, and the time for slow change of SZSE are all undergoing smooth changes.

In summary, the proposed MWMRQA approach can effectively capture the recurrence characteristics of multidimensional nonstationary nonlinear time series under different wavelet scales. It can clearly show the Lorenz system and stock market in what wavelet scale has strong determinism, complexity, and other recurrence characteristics. This will provide another perspective for researchers who want to study time series with higher complexity or deterministic, that is, a certain time series has higher deterministic or complexity at a certain wavelet scale, and then let them use the time series under this wavelet scale to carry out their own research. In further research, we will decompose the signal by processing non-stationary data according to the time-scale characteristics of the data itself, develop more multiscale methods, and explore more multiscale dynamic behaviors.

ACKNOWLEDGMENTS

We acknowledge financial support from the Scientific Research Common Program of Beijing Municipal Commission of Education (Grant No. KM.2018111232020) and the National Natural Science Foundation of China (NNSFC) (No. 61673005).

DATA AVAILABILITY

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to state restrictions such as privacy or ethical restrictions.

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