Multifractal Properties of Return Time Statistics

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The global statistics of the return times of a dynamical system can be described by a new spectrum of generalized dimensions. Comparison with the usual multifractal analysis of measures is presented, and the difference between the two corresponding sets of dimensions is established. Theoretical analysis and numerical examples of dynamical systems in the class of iterated functions are presented.

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Suppose that $T$ is a transformation of the space $X$ into itself, which preserves the probability measure $\mu$. Let $A$ be a measurable subset of $X$ of positive measure, and let $x$ be a point in $A$. Let also $\tau_A(x)$ be the (integer) time of the first return of $x$ to $A$:

$$\tau_A(x) = \inf\{n > 0 \text{ such that } T^n(x) \in A\}. \quad (1)$$

The Poincaré theorem guarantees that the return time is almost certainly finite with respect to any invariant measure. It can be rather long, and particularly short as well: If $x$ is a fixed point of $T$, then obviously $\tau_A(x) = 1$ for any set $A$ containing $x$. In this paper, we shall prove that the distribution of return times is characterized by multifractal properties which can be properly described by the tools of the thermodynamical formalism [1].

Much recent research has been focused on the local statistics of these returns. Let $B_r(x)$ be the ball of radius $r$ centered at a given point $x \in X$. Let $\lambda(r)$ be the measure of this ball. Next, consider the (normalized) cumulative distribution of the first return times, in the ball, of all points of the ball itself: $m(x,r,s) := \mu\{y \in B_r(x) \text{ s.t. } \lambda(r)T_{B_r(x)}(y) > s\}/\lambda(r)$. In many instances, these statistics become exponential: $m(x,r,s) \to e^{-s}$, as $r$ tends to zero, for $\mu$-almost all $x$. Rigorous proofs of this fact have been produced under an ever lessening set of chaoticity hypotheses [2].

Fundamental for our theory is the Kac theorem [3], a classical result of the local analysis: It predicts that, whenever the measure $\mu$ is ergodic with respect to $T$, for any measurable set $A$, the relative expectation of $\tau_A$ over the set $A$ is just the inverse of the measure of $A$:

$$\int_A \tau_A(y) \frac{d\mu(y)}{\mu(A)} = \frac{1}{\mu(A)}. \quad (2)$$

Furthermore, let $\mathcal{A}$ be a generating partition of $X$, and let us refine it around the point $x \in X$ by defining the cylinder of order $n$, $A_n(x)$, as the intersection of all the elements of $\mathcal{A}, T^{-1}\mathcal{A}, \ldots, T^{-n+1}\mathcal{A}$ containing $x$. The Ornstein-Weiss theorem [4] states that, whenever the measure $\mu$ is ergodic, the following limit exists $\mu$-almost everywhere, and is equal to $h(\mu)$, the metric entropy of $\mu$:

$$\lim_{n \to \infty} \frac{\log \tau_{A_n(x)}}{n} = h(\mu). \quad (3)$$

Parallel to the Ornstein-Weiss theorem, the Shannon-McMillan-Breiman theorem gives the metric entropy by means of the exponential decay of the measure of cylinders around almost all points. It is common in the physical usage to replace cylinders with balls: This is allowed in force of the Brin-Katok theorem [5]. Let us therefore substitute $A_n(x)$ and $n$ in Eq. (3) for $B_r(x)$ and $-\log r$, respectively. In the case of Gibbs measures of Axiom-A diffeomorphisms [6], and of a wide class of maps of the interval [7], the modified limit exists $\mu$-almost everywhere, and is equal to the information dimension $D_\mu(1)$ of $\mu$:

$$\lim_{r \to 0} \frac{-\log\tau_{B_r(x)}}{\log r} = D_\mu(1). \quad (4)$$

The definition of $D_\mu(1)$, and an informal proof of Eq. (4), will be given momentarily.

Thus far, the analysis has been local. Quite different—and more complex—is the case of the global statistics that we consider in this paper. We draw balls of
fixed radius $\varepsilon$ around any point of $X$, and we compute the integrals,
\[ \Gamma_\varepsilon(e, q) := \int_X \tau_{B_x}^{-q}(x) d\mu(x), \quad (5) \]
where $q$ is a real quantity. These are the statistical moments of the time required to come back to a neighborhood of the starting point: It is clear that local analysis alone has little to say about the scaling, in the radius $\varepsilon$, of these quantities.

Indeed, $\Gamma_\varepsilon(e, q)$ are a sort of partition functions, quite akin to those employed in the thermodynamical formalism. Their scaling for small $\varepsilon$ defines a new set of dimensions, $D_r(q)$:
\[ \Gamma_\varepsilon(e, q) \sim e^{D_r(q)(q-1)}, \quad (6) \]
which we call return time dimensions. These dimensions are the object of this Letter: To find their relations with the usual quantities of the thermodynamical formalism is the first question that demands an answer.

Let us start by observing that the return time $\tau_{B_x}(x)$ can be interpreted as a “single-point” sample of the integral in Kac theorem, Eq. (2). Therefore, one could estimate that
\[ \tau_{B_x}(x) \sim \mu(B_x(x))^{-1}. \quad (7) \]
It is worth recalling now that the local dimension $\alpha(x)$ of the measure $\mu$ at the point $x$ is defined by the scaling $\mu(B_x(x)) \sim e^{\alpha(x)}$, in the limit of small $\varepsilon$. Moreover, the relation $\alpha(x) = D_\mu(1)$ holds almost everywhere in $X$: This fact and Eq. (7) then imply the theorem expressed by Eq. (4).

The approximate equality (7) has been already adopted in [8,9] to evaluate, via Eq. (5), the exact thermodynamical partition function $\Gamma_\mu(e, q)$,
\[ \Gamma_\mu(e, q) := \int_X \mu(B_x(x))^q d\mu(x) \sim e^{D_\mu(q)(q-1)}, \quad (8) \]
whose scaling for small $\varepsilon$ gives the well-known spectrum of generalized dimensions $D_\mu(q)$ [10].

The substitution of Eq. (8) by Eqs. (5) and (6) is particularly significant, for it allows one to treat dynamical systems endowed with physical measures of the Sinai-Bowen-Ruelle type: In this case the integral can be replaced by a Birkhoff sum over the trajectory $x_l := T^l(x_0)$, $l = 0, \ldots$, of a generic point $x_0$:
\[ \Gamma_l(e, q) := \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \tau_{B_{x_j}}^{-q}(x_j). \quad (9) \]
Indeed, this Birkhoff procedure, Eq. (9), is the one originally employed in [8,9], where it is claimed to produce the spectrum of measure dimensions $D_\mu(q)$ [11]. A single sum is needed in Eq. (9), at difference with the technique of $q$-correlation integrals [12].

We are now ready to introduce the main result of this paper: Contrary to the usage of [8,9], the estimate (7) can be safely used only locally, or, as in Kac theorem, but not in the stronger sense required in Eqs. (5), and (6): The variables $\tau_{B_x}^{-1}(x)$ and $\mu(B_x(x))^{-1}$ have different large deviation properties. Therefore, $D_\mu(q)$ and $D_r(q)$ are not the same function, and the latter defines a new bona fide spectrum of dimensions.

The abstract proof of this statement would be scarcely informative, if not paralleled by a specific example. It is therefore convenient to introduce a family of dynamical systems whose invariant measures are completely known, in the sense that the spectrum of dimensions $D_\mu(q)$ can be easily and precisely computed: These are the so-called systems of iterated functions, or IFS [13]. In the simplest, one-dimensional case, a disconnected IFS consists of a collection of $M$ contracting maps $\phi_i$ of $[0, 1]$ to itself, such that $\phi_i[0, 1] \cap \phi_j[0, 1] = \emptyset$ for $i \neq j$. These maps can be thought of as the inverse branches of the dynamics: $(T \circ \phi_i)(x) = x$ for any $i$. In so doing, $T$ is characterized by a mixing repeller, that is, also the attractor of the collective action (in the sense that will be made soon clear) of the maps in the IFS.

A family of invariant measures for $T$ can be constructed assigning a probability, $\pi_i$, to each map $\phi_i$, $\sum_{i=1}^{M} \pi_i = 1$. The dynamics of IFS can be constructed by sequentially applying maps $\phi_i$, where $\sigma$ is chosen in a random fashion with probability $\pi_i \sigma$. Any orbit of the probabilistic IFS, when time reversed, becomes an orbit of the deterministic map $T$. Notice that return times are invariant under time reversal.

The reader will find useful to consider the usual ternary Cantor set measure as the invariant measure of the following two-maps IFS: $\phi_1(x) = x/3$, $\phi_2(x) = (x + 2)/3$.

The related invariant measure is an example of monofractal: All generalized dimensions are equal to the constant $\delta = \frac{\log 2}{\log 3}$. Let us perform the return time analysis of this dynamical system using Birkhoff sums, Eq. (9), and least squares fits over a finite range. Other techniques of estimating the integral (5) [14], and of extrapolating the exponent in (6), will be presented elsewhere.

We can first verify that the scaling relation (6) holds: Figure 1 assures us that this is indeed the case.

We can then extract the return dimensions $D_r(q)$ versus $q$: The numerical results are plotted in Fig. 2. They reveal a nontrivial range of dimensions: As far as return times are concerned, the dynamics are multifractal. Moreover, the return time dimensions are consistent with the constant value $\delta$ (within the statistical error bars) for negative dimensions, but are significantly different from this latter for positive values of $q$, the more, the larger the value of $q$.

Resorting now to exact analysis, we can confirm these results: Precisely, we are able to prove that (i) $D_\mu(0) = D_r(0)$, and (ii) $D_r(q) \to 0$ for large $q$. The second fact proves that the return dimensions $D_r(q)$ are significantly different from measure dimensions, while the first hints at relations that, at least in certain dynamical systems, may exist between them.
For a fixed value of $k$, let us choose $\varepsilon = \varepsilon_k = 1/3^k$. Then, simple geometric considerations assure us that, whenever $x$ belongs to $A^l_k$, with $k$ and $l$ fixed, it is also contained in the ball of radius $\varepsilon_k$ centered around any other point in $A^l_k$. In addition, these balls intersect no interval of generation $k$ other than $A^l_k$. This permits us to employ Kac theorem, Eq. (2), to compute exactly the sum $\Gamma_\tau(\varepsilon_k, 0)$:

$$
\Gamma_\tau(\varepsilon_k, 0) = \sum_{l=0}^{\infty} \int_{A^l_k} \tau_{A^l_k}(x) \, d\mu(x) = 2^k.
$$

(10)

It is then clear that (i) holds.

Remark that the intervals $A^l_k$ are also the cylinders of order $k$ of the dynamics. Therefore, the result we have just proven also applies to the return time dimensions, when defined via cylinders, and predicts that the dimension of order zero is equal to the topological entropy. Following the same idea, one also expects that the cylinder return time dimensions will be related to the Renyi entropies of the measure [15]. Many dimensional generalizations of this result are obviously possible, following the same geometric ideas.

It is evident that use of Kac theorem is permitted only for $q = 0$, for otherwise different fluctuation properties of the return time statistics set in. This leads us directly to the statement (ii). It is a standard observation that the return times define partitions of $X$ by the sets $R_n(\varepsilon) := \{x \in X \text{ s.t. } \tau_{B_n}(x) = n\}$. This allows us to rewrite the moments $\Gamma_\tau(\varepsilon, q)$ as

$$
\Gamma_\tau(\varepsilon, q) = \sum_{n=1}^{\infty} n^{-q} \mu[R_n(\varepsilon)].
$$

(11)

When $q$ is large, terms with small $n$ lead the sum (11). Let us retain just the first of these: It does not vanish by increasing $q$. This term is the measure of $R_1(\varepsilon)$, the set of points that move less than $\varepsilon$ in a single iteration of $T$. Clearly, all fixed points of $T$ belong to this set. Let $\bar{x}$ be any one of these, and let us assume that $\varepsilon$ is small enough, and that the transformation $T$ is smooth. Then, $R_1(\varepsilon)$ contains the ball of radius $\varepsilon/\Lambda$ centered at $\bar{x}$, where $\Lambda$ is the largest singular value of the matrix $1 - T'$. $1$ is the identity, and $T'$ is the Jacobian matrix of $T$ at $\bar{x}$. Moreover, the ball of radius $\varepsilon/\sigma$, where $\sigma$ is now the smallest singular value, contains the connected part of $R_1(\varepsilon)$ around $\bar{x}$. Then, we can estimate that, for large $q$,

$$
\Gamma_\tau(\varepsilon, q) = \sum_{\bar{x} \text{ s.t. } T(\bar{x}) = \bar{x}} \mu(B_{\rho \varepsilon}(\bar{x})),
$$

(12)

where the sum is extended to all fixed points of $T$, and where $\rho$ is a multiplicative factor, which depends on $\bar{x}$: $\sigma < \rho < \Lambda$. When the number of terms in the sum is finite, the leading contribution to (12) comes from the fixed point $\bar{x}$ with the smallest local dimension, $\alpha_m$: Since $\mu(B_{\varepsilon}(\bar{x})) \sim \varepsilon^{\alpha_m}$, it follows at once that

$$
D_\tau(q) = \frac{\alpha_m}{q - 1}, \quad \text{when } q \to \infty.
$$

(13)
We can readily verify that Eq. (13) is satisfied by the Cantor set dynamics introduced above. The fixed points of $T$ are here zero and one, and they are both characterized by the same value of the local dimension, $\alpha_m = \frac{\log 2}{\log 3}$. In Fig. 2, the relation (13) is validated by the numerical results of $D_t(q)$ for large $q$.

This investigation is not limited to a single dimension. We now add a two-dimensional example with a nontrivial dimension function $D_\mu(q)$: The motion on a Sierpinski gasket, corresponding to the IFS maps $\phi_1(x,y) = (x/2, y/2), \phi_2(x,y) = [(x+1)/2, (y+1)/2], \phi_3(x,y) = [x/2, (y+1)/2]$ on $[0,1]^2$, with nonuniform probabilities $\pi_1 = 4/10$, $\pi_2 = \pi_3 = 3/10$. In Fig. 3, we have compared the exact thermodynamical function $D_\mu(q)$ and the numerically evaluated $D_t(q)$. For negative values of $q$, the two dimensions are very close, and discrepancies may be due to the finite length of the Birkhoff sum, and, more importantly, of the fitting interval in $\varepsilon$, which cannot be accounted for by the statistical definition of the error bars. For positive values of $q$, to the contrary, the two dimensions are rather different, and the asymptotic formula (13) soon becomes a good approximation of $D_t(q)$.

In conclusion, we have shown that the statistics of return times are characterized by multifractal properties, well defined by a new spectrum of dimensions $D_t(q)$. We have also found that, contrary to implicit previous statements in the literature, this spectrum does not coincide with the usual multifractal spectrum $D_\mu(q)$. It is now interesting to understand the meaning of the Legendre transforms associated to these new dimensions. We have proven that $D_t(0) = D_\mu(0)$ under certain hypotheses on the dynamical system. Approximate equality seems to hold for negative $q$’s as well. This might provide a means for computing negative measure dimensions, which is known to be a challenging numerical task. Finally, we have shown that in a large class of systems the relation $D_t(q) = \frac{\alpha_m}{q}$ holds for large $q$, where $\alpha_m$ is the smallest local dimension at the fixed points of $T$.

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[11] After this paper had been submitted, we received a communication from Professor Grassberger indicating that the actual numerical computations for [8] were performed by Birkhoff sums of the kind $\sum_{i,j} \tau_{B_{\ell}(x_i)}(x_j)$. We shall comment elsewhere on the effect of this choice.


[15] If one considers cylinders in place of balls, the two generalized dimensions can be identified with the deviation functions of the variables $\tau_{A_{\ell}(x)}(x)$ and $\mu(A_{\ell}(x))$ [16]. Then, one can rigorously prove that the two dimensions coincide in a small neighborhood to the right of $q = 1$, but they must differ for large $q$ [17].
