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Coupling analysis and the problem of time-delayed embedding

Andreas Groth
Ernst-Moritz-Arndt-University of Greifswald, Jahnstrasse 15a, 17487 Greifswald, Germany
(Dated: October 15, 2007)

Reconstructing properties of dynamical systems from a time delayed embedding of observed time series is known to be very sensitive to the time delay and the embedding dimension. Depending on the problem there are various techniques to find proper embedding parameters. Here we study the impact of the time delay to a bivariate coupling analysis by means of order recurrence plots. On coupled self-sustained chaotic oscillators we demonstrate the problem of a successful detection of synchronization, where the choice of proper time delays is examined in comparison to common approaches of mutual information and replacement from diagonal. This paper gives more insight into the recently introduced concept of a recurrence analysis on ordinal scale and the impact of embedding parameters.

PACS numbers: 05.45.Tp, 05.45.Xt

I. INTRODUCTION

In the analysis of coupled dynamical systems various techniques have been developed to detect dependencies from observed time series. Based on the nature of studied systems, there are different conceptual ideas. The coupling analysis in this paper is motivated by a fundamental property of dynamical systems: recurrence. This property has been first pointed out by Poincaré [1]. For an overview about synchronization analysis of complex systems by means of recurrence and recent developments we refer to [2]. In order to develop methods for practical time series with just a few hundreds or thousands values the recurrence of states in phase space has to be considered on a coarser-grained level. But instead of measuring distances between states, we define recurrence just by the order structure of state vectors. The first case we refer to as recurrence on metric scale and the latter case as recurrence on ordinal scale [3]. In view of an application to experimental data an analysis on ordinal scale has the main advantage to be more robust with respect to non-stationarities, which is invariant with respect to an arbitrary monotonic transformation of the values of a time series.

In this paper we discuss the proper choice of embedding parameters and their delay to our coupling analysis. In detail we consider a reconstruction of dynamical systems behavior via time delayed embedding, where we have two main questions: On the one hand there is the choice of dimension. A general access to this problem can be found in the embedding theorem of Takens [4, 5], which gives an idea of sufficient dimensions. Beside this theoretical approach there are several numerical methods to derive directly from the data necessary dimensions (see e.g., [6] and reference within). On the other hand there is the choice of time delay. Here, Takens embedding theorem states that almost any time delay is acceptable. But this is not very useful when considering practical data with finite resolution and measurement errors. There are several concepts to find proper time delays (auto-correlation [7], mutual information [8], correlation integral [9, 10], geometry-based methods [11]). But all these concepts consider recurrence on metric scale and we demonstrate that this is not always sufficient to find proper parameters in the ordinal case. Moreover, we motivate the relevance and necessity of a self-consistent analysis on ordinal scale.

As a prototypical model of coupling we study the phenomenon of synchronization of self-sustained oscillators. The recurrence behavior is related to the dynamical properties of systems and the coupling strength is reflected in the properties of recurrence, such as the recurrence rate. We illustrate the necessity of proper embedding parameters to reveal synchronization, especially in case of noise corruption.

This paper is organized as follows. First we review the concept of recurrence on ordinal scale. Then we discuss embedding problems and derive quantities to assess embedding parameters. Finally, in a numerical application to coupled chaotic oscillators we evaluated the choice of proper time delays.

II. RECURRENCE ANALYSIS ON ORDINAL SCALE

To understand real world experiments there are various ideas to model the evolution in time. In the formalism of dynamical systems the behavior is described by a trajectory

$$\vec{x}(t) = [x_1(t), x_2(t), \ldots, x_d(t)] \in \mathbb{R}^d,$$

where $x_n(t)$ denotes the $n$-th component at time $t$. In the case of continuous time the evolution from current to future states is modeled by a flow $F$. Later on, we consider self-sustained oscillators where the dynamics is described by differential equations $\dot{\vec{x}} = F(\vec{x})$. The component $x_n$ is an accessible physical value, such as the position, speed,
and velocity of a particle. However, in an experimental setting not all relevant components are typically known or can be measured. A common concept to reconstruct the dynamics from a scalar measurement \( u(t) = f(x(t)) \) is the time-delayed embedding [4, 5]

\[
\mathcal{U}(t) = [u(t), u(t+\vartheta), \ldots, u(t+(m-1)\vartheta)] \in \mathbb{R}^m.
\] (2)

Here we simply choose the first component \( u(t) = x_1(t) \).
In order to get time series from time-continuous model, we write \( x_i = x(i\Delta t) \) and \( \mathcal{U}_i = \mathcal{U}(i\Delta t) \), respectively, where \( i = 1 \ldots N \) and \( \Delta t \) is the sampling period.

Intuitively, recurrence means repetition, where we look for similar states of \( x \) or \( \mathcal{U} \). In case of finite data sets maybe corrupted by noise, an exact recurrence \( x_i = x_j \) is not relevant. To this reason we just require that \( x_i \) is in the neighborhood of \( x_j \): \( ||x_i - x_j|| \leq \varepsilon \). The threshold \( \varepsilon \) defines the coarse graining level, where the choice is essential for the analysis. Among others it is important to find a so-called scaling region.

In the case of a recurrence analysis on ordinal scale we circumvent this problem. Here we neglect distances and define recurrence on ordinal scale if two states \( \mathcal{U}_i \) and \( \mathcal{U}_j \) exhibit the same order structure \( \pi(\mathcal{U}_i) = \pi(\mathcal{U}_j) \). The order structure is encoded by \( \pi \), also denoted as order pattern. To explain this idea, let us start with embedding dimension \( m=2 \). Two relations

\[
\pi = \begin{cases} 
1, & u_i < u_{i+\vartheta}, \\
2, & u_i > u_{i+\vartheta}, 
\end{cases}
\]

are possible, which decompose the reconstructed phase space into two areas, separated by the main diagonal.\(^3\)

This way of deriving binary symbol sequences is not very origin and quite common in different fields. But this idea can be extended to higher dimensions. Next, at embedding dimension \( m=3 \), six order patterns are possible, which decompose the phase space into six equivalent regions (Fig. 1). These regions are separated by three planes of pairwise equality of the components \( u_k = u_{k+\vartheta} \), \( u_k = u_{k+2\vartheta} \), \( u_k = u_{k+3\vartheta} \), and \( u_k = u_{k+2\vartheta} \) and \( u_k = u_{k+3\vartheta} \). The successive oscillation through the order patterns in Fig. 1 suggests a connection to the phase of an oscillator. Indeed this has mathematically been shown in [3].

In general, the phase space \( \mathbb{R}^m \) is decomposed into \( m! \) regions. Originally, this idea of a decomposition is motivated by the definition of permutation entropy [12]. In order to find proper embedding parameters, we are able to assess the decomposition of trajectories visually at low dimensions and small data sets. But at higher dimensions or extensive data sets a quantitative rating is necessary.

\[\text{FIG. 1: (a) Time-delayed embedding of the first component of a Rössler system (6) with embedding dimension } m=3 \text{ and time delay } \vartheta=2. \text{ Phase space vectors in the same region are considered as recurrent (black circles), and in different regions are not recurrent (empty circle). (b) Same plot with view angle in direction of main diagonal.}\]

A. Coupling analysis

In order to detect coupling between two dynamical systems we analyze cooperative behavior by means of cross recurrence [13, 14]. Suppose we observe a second dynamical system \( \mathcal{Y} \) with its time-delayed embedding \( \mathcal{U}_i = (v_i, v_{i+\vartheta}, \ldots, v_{i+(m-1)\vartheta}) \). Then, cross recurrence is defined as \( \pi(\mathcal{U}_i) = \pi(\mathcal{U}_{i+\tau}) \). As a helpful tool for a first visual inspection of coupling we consider the order recurrence plot

\[
R_{i,\tau} = \begin{cases} 
1, & \pi(\mathcal{U}_i) = \pi(\mathcal{U}_{i+\tau}), \\
0, & \text{otherwise},
\end{cases}
\] (3)

\[\text{\footnotesize\textit{Footnote}}\]

\(^3\) In general we neglect the equality of values. This is reasonable if we consider systems with continuous distribution of the values, where equality has measure zero. In the numerical applications equality has not been separately considered, here we only test for \( < \) and \( \geq \).
as an analogy to the recurrence plot [15]. There are various quantities to determine different properties of dynamical systems. In order to detect coupling we determine the recurrence rate

\[ r_\tau = \frac{1}{L_\tau} \sum_i R_{i,\tau} \]

as a function of the time lag \( \tau \) between both observed systems. The recurrence rate is normalized by the respective number of elements \( L_\tau \) in the \( \tau \)-th row of \( R \), thus, we get \( 0 \leq r_\tau \leq 1 \). To get a single coupling index from the recurrence rate we determine the maximal value

\[ r_{\text{max}} = \max_\tau \{ r_\tau \}. \]

To demonstrate this concept we consider two diffusively coupled Rössler systems

\[
\begin{align*}
\dot{x}_1 &= -\omega_1 x_2 - x_3 + k (y_1 - x_1) \\
\dot{y}_1 &= -\omega_2 y_2 - y_3 + k (x_1 - y_1) \\
\dot{x}_2 &= \omega_1 x_1 + a x_2 \\
\dot{y}_2 &= \omega_2 y_1 + a y_2 \\
\dot{x}_3 &= b + x_3 (x_1 - c) \\
\dot{y}_3 &= b + y_3 (y_1 - c)
\end{align*}
\]

with some parameters \( a, b \) and \( c \). With these parameters we are able to change the typical recurrence behavior. In this paper we consider two essentially different regimes of chaotic oscillation, a phase-coherent regime \( (a=0.15, b=0.2, c=10) \) and a non-phase-coherent regime \( (a=0.28, b=0.1, c=8.5) \). The oscillators are detuned \( \omega_1 = 0.95 \), \( \omega_2 = 0.99 \), and the equations are numerically solved with \( \Delta t = 0.1 \). If the coupling strength \( k \) is strong enough we observe a locking of the oscillations, although the amplitudes remain uncorrelated [16]. However, the order patterns are connected to the phase, consequently, the phase synchronization is reflected in the order recurrence plot in Fig. 2(a-b). The locking of oscillations causes horizontal lines, which in turn results in distinct peaks in the recurrence rate \( r_\tau \). Moreover, the plot reveals very well the differences between phase- and non-phase-coherent regime. In the phase-coherent regime there is a whole set of horizontal lines, as a result of a narrow distribution of recurrence times [Fig. 2(a)]. In the non-phase-coherent regime the distribution is broad, hence, we essentially observe only a single long line at \( \tau \approx 0 \) [Fig. 2(b)]. In the case of independent random processes the order recurrence plot shows a random structure, too [Fig. 2(c)], and the recurrence rate is flat with \( r_\tau \approx 1/6 \). In general we have at dimension \( m \) an expectation value of \( 1/m! \).

III. OPTIMAL TIME DELAY

Now we demonstrate the necessity of a proper choice of the time delay to successfully detect coupling. Small delays cause approximately equal values of the components of a state vector. Hence, the trajectory is compressed along the main diagonal, what increases the sensitivity of the analysis to noise. Larger delays yield to more unrelated components, what unnecessarily complicates the behavior of the trajectory and the analysis of recurrence. These effects are also denoted as redundancy and irrelevancy error, respectively [17]. On metric scale there are numerous concepts to find a proper balance. For a comparison we select on the one hand the concept of mutual information [8], where the transition from dependent to independent components is measured. On the other hand we select the replacement from diagonal [11], where the expansion of trajectories in phase space is determined.

A. Motivation of a self-contained analysis

Considering the two Rössler oscillators from Eq. (6), we observe the well-known phenomenon of phase synchronization [18], what can be described in terms of classical phase locking [16]. The locking of both phases is also captured by the cross recurrence of order patterns and is reflected in a saturation of the recurrence rate (Fig. 3). The dashed line indicates the onset of phase synchronization, where a zero Lyapunov exponent becomes negative. Results for multiple embedding parameters are shown, and it clearly turns out that the time delay is a crucial factor to detect phase synchronization, especially in the noise-corrupted case. In the non-phase-coherent regime we obtain a similar picture (not shown here), but with a saturation at much higher coupling strength, where in complete agreement with [19] a locking of phases is observed beyond generalized synchronization \( (k \geq 0.2) \).

To rate the quality of synchronization detection we consider the difference in the coupling index

\[ \Delta r_{\text{max}} = r_{\text{max}} (k = k_0) - r_{\text{max}} (k = 0) \]

\[ \Delta r_{\text{max}} = r_{\text{max}} (k = k_0) - r_{\text{max}} (k = 0) \]
between uncoupled $k=0$ and synchronized oscillators $k=k_0$ as a function of the time delay $\vartheta$. In the phase-coherent and non-phase-coherent regime we choose $k_0=0.05$ and $k_0=0.25$, respectively.

To motivate a self-consistent analysis of the embedding parameters completely on ordinal scale we compare $\Delta r_{\text{max}}$ with two metric approaches: the replacement from diagonal (RFD) and mutual information (MI) (Fig. 4). Despite some correlations in the general behavior between $\Delta r_{\text{max}}$, RFD, and MI, there are a lot of discrepancies. They are most significant in embedding dimension $m=2$. Here the quality of coupling detection $\Delta r_{\text{max}}$ is nearly independent from the time delay over a large range, what is contradictory to RFD, MI, and the general view of a trajectory expansion in phase space. Even in the case of a moderate level of noise the impact of the time delay can be neglected, if $\vartheta$ is smaller than a mean period of the oscillator. At higher dimensions the time delay becomes more important, what requires a careful choice. The quality of coupling detection $\Delta r_{\text{max}}$ decreases every time, when two or more components in $\vartheta$ or $\vartheta$ have a distance of a mean period $T \approx 6.3$. In this case the trajectory is compressed in the corresponding subspace and even small perturbations can corrupt the order patterns and with it the coupling analysis. This embedding problem results from the special way of a de-

composition by order patterns and cannot be completely captured by RFD, MI or any other metric measure of expansion. To this reason we advance a self-consistent analysis and discuss embedding problems directly on ordinal scale. Moreover, it is obvious to analyze the already present sequences of order patterns, instead of calculating additional metric measures.

B. Permutation entropy and mean interval length

Instead of performing the coupling analysis on a large range of time delays, we now motivate two quantities to rate the quality of embedding and draw conclusion to a successful coupling analysis. These quantities take into account statistical and dynamical properties. However, it is obvious that just from the analysis of a single system, we can not definitely conclude to the coupling analysis, but with the following tools we are able to pre-select proper ranges in a very fast way. This pre-selection algorithm is typically three orders of magnitude faster than the full coupling analysis.

As a first parameter we consider the distribution of the trajectory within the regions of order patterns. Given the frequency a certain order pattern appears $p_n = \text{prob}(r(\tilde{u}_i) = n, i = 1, \ldots, N)$ we analyze the expansion of the trajectory by means of the Shannon entropy

$$H = - \sum_{n=1}^{m!} p_n \log p_n. \quad (8)$$

This quantity has already been introduced as a complexity measure and is referred to as permutation entropy [12]. There are two extrema. On the one hand, when the trajectory passes all order patterns with the same probability (uniform distribution), the permutation entropy becomes maximal with $H = \log m!$. All other distributions cause smaller values, and we get $H = 0$, if only one order pattern is observed. To this reason we are able to rate the expansion of the trajectory in phase space by $H$. Further on, we normalize $H$ by

$$h = H / \log M, \quad (9)$$

where $M \leq m!$ is the number of observed order patterns, and $0 \leq h \leq 1$. In contrast to [12] we do not normalize by $\log m!$ to take into account a possible restriction of the trajectory to a subset of order patterns. This is typical for higher embedding dimensions and simple trajectories, where a normalization by $\log M$ acts as a penalty function, to avoid unnecessarily complicated behavior at large time delays. Nevertheless, $h$ takes into account only statistical properties and does not distinguish between expansion due to noise or enfolding. Especially at $\vartheta \approx T \approx 6.3$ the permutation entropy suggest a perfect distribution of the trajectory (Fig. 5), where we have $h \approx 1$ at all dimensions. In all other cases $h$ gives some reliable drawbacks to proper time delays. However,
we see that from the distribution of order patterns we are not able to capture the impact of noise. To handle this problem we additionally consider dynamical properties.

We assume that the underlying dynamical systems generates a smooth trajectory, corrupted by a moderate level of observational noise. To assess the quality the sequence of order patterns captures the relevant smooth behavior, we consider the interval length \( L \), the trajectory spends in a region

\[
\pi(\tilde{u}_{i-1}) \neq \pi(\tilde{u}_i) = \cdots = \pi(\tilde{u}_{i+L-1}) \neq \pi(\tilde{u}_{i+L}).
\]  

(10)

From the time series we get a set of \( k \) disjunct intervals of constant order patterns, where each interval has length \( L_k \). Then, we consider as a second quantity the mean interval length

\[
\bar{L} = \frac{1}{K} \sum_{k=1}^{K} L_k,
\]  

(11)

which gives information about the smoothness of the trajectory. In contrast to the permutation entropy this function rejects \( \vartheta \approx T \approx 6.3 \) (Fig. 5). Moreover the impact of increasing noise is well captured. For fixed dimension \( m \) and varying time delay \( \vartheta \) we expect to minimize fluctuations errors of the trajectory between order patterns by maximizing \( \bar{L} \).

Finally, we combine both quantities into a single index

\[
q = q(\vartheta) = h \cdot \bar{L}
\]  

(12)

to find proper time delays. In case of the phase-coherent Rössler system with a clear dominant oscillation, \( q \) gives some reliable information about problematic time delays (Fig. 5, lower panel), which cannot be derived from RFD or MI.

Next we demonstrate, that just from the analysis of embedding quality of a single system we cannot always directly conclude to the quality of coupling detection. We consider two non-phase-coherent Rössler systems in generalized synchronization. The trajectories are more complicated with a broad distribution of return times, what results in less dominant peaks in \( q \) and RFD (Fig. 6). Nevertheless, all quantities of embedding quality show clear variations and suggest time delays. But this is contrary to the quality of coupling detection \( \Delta r_{\text{max}} \) which is nearly independent from the time delay (Fig. 6, upper panel). In the present case of almost identical systems we observe lag synchronization [20], where we have \( u_i \approx u_{i+\tau} \). Due to this strong relationship the recurrence rate becomes almost everywhere maximal, except small time delays, where the relationship is hidden by noise. This discrepancy between \( q \) and \( \Delta r_{\text{max}} \) becomes less dominant with increasing noise level, where the strong relationship \( u_i \approx u_{i+\tau} \) is hidden and only revealed at
proper time delays.

IV. SUMMARY

We conclude that the choice of time delay in the concept of time delayed embedding is crucial to the analysis of coupling. This problem as already been widely discussed in the context of dimension estimation of attractors. In different papers it has been concluded, that a pre-selection of time delays is not generally successful and a detailed estimation of the dimension is necessary. Here, we confirm this problem in the context of a coupling analysis. Although we could present some analogies between embedding quality and quality of coupling detection, a detailed analysis of coupling for multiple parameter settings is necessary, too. However, the present measure of embedding quality offers a fast pre-selection tool.

Acknowledgments

The author would like to thank C. Bandt for fruitful discussions that have helped to write this paper. This work was supported by the priority programme SPP1114 of the German Research Foundation.

FIG. 6: Same coupled systems as in Fig. 4 but additionally with permutation entropy $h$, mean interval length $L$, and embedding quality index $q$.

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