STRONG LAWS FOR RECURRENCE QUANTIFICATION ANALYSIS

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The recurrence rate and determinism are two of the basic complexity measures studied in the recurrence quantification analysis. In this paper, the recurrence rate and determinism are expressed in terms of the correlation sum, and strong laws of large numbers are given for them.

Keywords: Strong law of large numbers; recurrence quantification analysis; recurrence rate; determinism; correlation integral.

1. Introduction

The notion of recurrence is one of the fundamental notions in the theory of dynamical systems. Recurrence plots, introduced by Eckmann et al. [1987], provide a powerful tool for recurrence visualization. The recurrence plot of the trajectory \( x_0, x_1, \ldots, x_{n-1} \) of a point \( x = x_0 \) is a black-and-white image with a pixel \((i, j)\) being black if and only if the trajectory at time \( j \) recurs to the state at time \( i \); that is, the points \( x_i, x_j \) are close to each other. The recurrence plot provides a two-dimensional representation of an (arbitrary-dimensional) dynamical system.

The quantitative study of recurrence plots, called recurrence quantification analysis (RQA), was initiated by Zbilut and Webber [1992], where the authors introduced several measures of complexity based on the recurrence plot. Among them, the recurrence rate \( RR \) and the determinism \( DET \) are probably the most important and widely used ones. Their definitions are based on diagonal lines (that is, segments of black points parallel to the main diagonal), which correspond to recurrences of parts of the trajectory.

Since the seminal paper by Zbilut and Webber [1992], new RQA tools, quantities and modifications were introduced and recurrence quantification has been applied in many areas of science, cf. [Marwan et al., 2007] and [Kulkarni et al., 2011], among others.

Despite its wide use, theoretical properties of recurrence measures were studied rarely. Asymptotic properties of RQA characteristics were studied e.g. in [Faure & Korn, 1998; Thiel et al., 2003; Zou et al., 2007; Faure & Lesne, 2010; Donges et al., 2012]; see also [Robinson & Thiel, 2009]. The correlation sum, tightly connected with the recurrence rate, as well as derived quantities such as the correlation integral, correlation dimension and correlation entropy, were studied extensively, cf. [Kantz & Schreiber, 2004]. One of the fundamental results, namely the strong law for correlation sums of ergodic processes, was proved (by different methods and under different conditions) in [Pesin, 1993; Pesin & Tempeleman, 1995; Aaronson et al., 1996; Serinko, 1996; Manning & Simon, 1998]. It states that, for a separable metric space \((Z, d)\) and a \(\mu\)-ergodic dynamical system on it, the correlation
sum of the trajectories of almost every point \( x \in Z \)
with every (up to countably many) distance threshold \( r > 0 \) converges to the correlation integral. That is,
\[
\lim_{n \to \infty} C(x, n, r) = c(r), \quad \text{where}\]
\[
c(r) = \mu \times \mu((x, y) : d(x, y) \leq r)
\]  
(1)
and \( C(x, n, r) = (1/n^d) \cdot \text{card}((i, j) : 0 \leq i, j < n, \ d(x_i, x_j) \leq r) \). The correlation integral \( c(r) \) is a highly nontrivial characteristic of the measure \( \mu \).

It is worth noting that the correlation sum, which measures the level of dependence in a trajectory, asymptotically turns into the probability of closeness of two independent random variables.

The main purpose of the present paper is to study asymptotic properties of RQA characteristics for ergodic processes. The results apply equally well to the ergodic dynamical systems. We start with a proof of a simple formula giving an expression of the recurrence rate via correlation sums, see Proposition 1:
\[
\text{RR}^m_k = k \cdot C^m_k - (k - 1) \cdot C^m_{k+1},
\]  
(2)
where \( m \) is the embedding dimension, \( k \) is the prediction horizon and \( C^m_k \) and \( \text{RR}^m_k \) denote the correlation sum and recurrence rate, respectively; for the corresponding definitions see Sec. 2. Notice that the quantities depend also on \( x, n, r \), though the dependence is not made explicit. The relationship (2) directly permits to express the determinism \( \text{DET} \) in terms of the correlation sums:
\[
\text{DET}^m = \frac{\text{RR}^m}{\text{RR}^1} = k \cdot C^m_k - (k - 1) \cdot C^m_{k+1},
\]  
(3)
The bridging formulas (2) and (3) enable us to derive strong laws of large numbers for the recurrence rate and determinism from that for the correlation sum, see Theorems 4 and 5. To this end, however, we need to generalize (1) to the case when \( d \) is a pseudometric on \( Z \), rather than a metric; i.e. the distance between distinct points may vanish. For pseudometrics induced by Borel maps, this problem was studied by Serinko [1996, Theorem 2]. In the general case, (1) was proved by Manning and Simon [1998]; for details, see Theorem 19 in Sec. 7.

We apply the strong laws to iid processes, Markov chains and autoregressive processes, and derive explicit formulas for the recurrence integral, asymptotic determinism and mean diagonal line length of these processes; see Table 1 and Sec. 4. On simulated data we demonstrate the speed of convergence of RQA quantities when the length \( n \) of (the beginning of) the trajectory goes to infinity.

In Sec. 5 we give an example showing that the higher metric entropy of a process does not necessarily mean smaller (asymptotic) determinism and that an iid process can have higher determinism than a non-iid one with the same one-dimensional marginals. This is a rather unexpected behavior since, in a sense, the metric entropy and determinism are opposite notions.

We also discuss the problem of choosing the distance threshold \( r \). In the literature, the distance threshold is selected such that, for the embedding dimension \( m \), the recurrence rate attains a fixed level. This rule, however, can lead to the existence of the so-called spurious structures in recurrence plots of iid processes, as noted by Thiel et al. [2006], see also Marwan et al. [2007, Section 3.2.4]. In Sec. 6 we show why this happens. For a large embedding dimension \( m \) and distance threshold \( r_m \) selected by this rule, the determinism \( \text{DET}^m(r_m) \) is close to one even for iid processes. Hence, the appearance of spurious structures is a direct consequence of the selection rule fixing the recurrence rate.

The explicit formula for the asymptotic determinism \( \text{DET}^m \) can be stated in terms of conditional probabilities that \( k \) or \( k + 1 \) consecutive recurrences occur given that one recurrence has occurred; see Theorem 10. Hence, if the process under consideration is a Markov one of order \( p \), then over-embedding to dimension \( m \geq p \) leaves

| Table 1. Formulas for RQA asymptotics for iid processes and Markov chains. Here \( a = c(r) \) and \( \beta = c_{11}(v) \), see (16). |
|-----------------|-----------------|-----------------|
| IID | Markov Chain |
| Recurrence integral \( a^{k+1}([k - (k - 1)]a] \) | \( a_{i,j}^{k+1}([k - (k - 1)]a] \) |
| Asymptotic determinism \( a^{k+1}([k - (k - 1)]a] \) | \( \beta^{k+1}([k - (k - 1)]a] \) |
| Mean diagonal line length \( k + a/1 - a) \) | \( k + \beta/(1-\beta) \) |
the asymptotic determinism unchanged; see Corollary 11. This is demonstrated in Sec. 4.3 for autoregressive processes. There we discuss possible use of RQA characteristics for estimation of the order of such processes.

The paper is organized as follows. In Sec. 2 we recall the definitions of RQA measures and we prove (2), see Proposition 1. The strong laws are stated in Sec. 3 and, in Sec. 4, they are applied to iid processes, Markov chains and autoregressive processes. Relationship between the metric entropy and asymptotic determinism is discussed in Sec. 5 and the so-called spurious structures in recurrence plots of iid processes are explained in Sec. 6. In Sec. 7 we discuss the strong law for correlation sums on pseudometric spaces.

2. Recurrence Quantification Analysis (RQA) and Correlation Sums

In this section we recall the definitions of basic RQA measures and of the correlation sum for (embedded) trajectories of a general S-valued process. To make the notation easier, we write \( x_{0}^{m} \), \( x_{1}^{m} \) as a shorthand for \((x_{0},x_{1},...,x_{m})\) and \((x_{0},x_{1},...,x_{m-1})\) respectively.

Let \( S = (S,d) \) be a metric space. Fix an integer \( m \geq 1 \) called the embedding dimension. Let \( S^{m} \) be the embedding space of \( m \)-tuples \( s_{0}^{m} \) equipped with a metric \( \rho^{m} \) compatible with the product topology. Natural choices for \( \rho^{m} \) are the Manhattan (\( L_{1} \)), Euclidean (\( L_{2} \)) or Chebyshev (\( L_{\infty} \)) metrics, the latest given by

\[
\rho^{m}(s_{0}^{m},t_{0}^{m}) = \max_{0 \leq j < m} \rho(s_{j}, t_{j}),
\]

but in general we do not restrict \( \rho^{m} \) to be one of these.

Let \( S^{\infty} \) denote the space of all sequences \( x_{0}^{\infty} \) of points from \( S \). This product space is usually equipped with a metric, say with \( \rho^{\infty}(x_{0}^{\infty},y_{0}^{\infty}) = \sum_{i} 2^{-i} \cdot \min\{1, \rho(x_{i}, y_{i})\} \). In practice, however, we know just (finite) beginnings of trajectories \( x_{0}^{m} \) and thus we are not able to compute the distance exactly. That is why we use pseudometrics, depending only on the first members of sequences. For an integer \( k \geq 1 \), called the prediction horizon, a pseudometric \( d_{k}^{m} \) on \( S^{\infty} \) is defined by

\[
d_{k}^{m}(x_{0}^{\infty},y_{0}^{\infty}) = \max_{0 \leq j < k} \rho^{m}(x_{j}^{\infty},y_{j}^{\infty}).
\]

For \( k = 1 \) we write simply \( d^{m} \) instead of \( d_{1}^{m} \). Notice that \( d_{k}^{m} \) depends only on the first \((m+k-1)\) members of \( x_{0}^{\infty},y_{0}^{\infty} \).

2.1. RQA measures

Fix a sequence \( x = x_{0}^{\infty} \in S^{m} \) and consider the embedded trajectory \( \tilde{x} = \tilde{x}_{0}^{\infty} \), \( \tilde{x}_{i} = x_{0+i}^{m} \in S^{m} \). Fix also a distance threshold \( r \geq 0 \). For \( i,j \in N \) (here \( N \) stands for the set of non-negative integers \( \{0,1,2,\ldots\} \)) we say that the couple \((i,j)\) is an \( r \)-recurrence (in the \( m \)-th embedding of the trajectory of \( x \)) if

\[
d^{m}(x_{i}^{m}, x_{j}^{m}) = \rho^{m}(x_{i}^{m}, x_{j}^{m}) \leq r.
\]

The recurrence plot of dimension \( n \) is a square \( n \times n \) matrix of zeros and ones, with the entry at \((i,j)\) equal to one if and only if \((i,j)\) is a recurrence. Usually, the recurrence plot is visualized by a black-and-white image, with black pixels representing recurrences. Let us note that to construct the \( n \times n \) recurrence plot (in the \( m \)-th embedding) one needs to know only the first \((n+m-1)\) members \( x_{0}^{m-1} \) of \( x \).

Diagonal lines are basic patterns in the recurrence plot. We say that \((i,j)\) is a beginning of a diagonal line of length \( k \geq 1 \) in the \( n \times n \) recurrence plot if the following are true:

- \( 0 \leq i, j \leq n-k \);
- \((i+h,j+h)\) is a recurrence for every \( 0 \leq h < k \);
- either at least one of \( i, j \) is equal to \( 0 \) or \((i-1,j-1)\) is not a recurrence;
- either at least one of \( i+k, j+k \) is equal to \( n \) or \((i+k,j+k)\) is not a recurrence.

For \( 0 < i, j < n-k \) this is equivalent to

\[
d_{k}^{m}(x_{i}^{m}, x_{j}^{m}) \leq r,
\]

\[
d_{k}^{m}(x_{i-1}^{m}, x_{j-1}^{m}) > r \quad \text{and} \quad d_{k}^{m}(x_{i+k}^{m}, x_{j+k}^{m}) > r.
\]

The number of lines of length \( k \) in the \( n \times n \) recurrence plot is denoted by \( L_{k}^{m}(x,n,r) \). Notice that the main diagonal line (i.e. the case \( i = j \)) is not excluded, thus \( L_{0}^{m}(x,n,r) = 1 \); further, \( L_{k}^{m}(x,n,r) = 0 \) for every \( k > n \).

Now fix the prediction horizon \( k \geq 1 \). The \( k \)-recurrence rate \( RR_{k}^{m} \) is the percentage of recurrences contained in diagonal lines of length at
least \( k \); that is,
\[
RR^m_n = RR^m_n(x, n, r) = \frac{1}{n^m} \sum_{i,j} l \cdot L_i^n. \tag{6}
\]
The \( k \)-determinism \( DET^m_n \) is the ratio of the \( k \)-recurrence rate and \( k \)-recurrence rate
\[
DET^m_n = DET^m_n(x, n, r) = \frac{RR^m_n}{RR^m_1} \tag{7}
\]
(here and throughout we always assume that the denominator is nonzero; otherwise we leave the corresponding quantity undefined). The \( k \)-average line length \( LAVG^m_n \) is the average length of diagonal lines not shorter than \( k \)
\[
LAVG^m_n = \frac{RR^m_n}{\frac{1}{n^m} \sum_{i,j} L_i^n}. \tag{8}
\]
again, this characteristic depends also on \( x, n, r \). For the definitions of other RQA characteristics, such as the (Shannon) entropy of diagonal line length, trend or measures based on vertical lines, see e.g. [Marwan et al., 2007].

2.2. Correlation sum

Tightly connected with the recurrence rate is the notion of correlation sum, studied by Grassberger and Procaccia [1983a, 1983b] in relation to the correlation dimension. For a sequence \( x = x_0^n \in S^\infty \), the embedding dimension \( m \), the prediction horizon \( k \), the distance threshold \( r \geq 0 \) and \( n \geq 1 \), the correlation sum is defined by
\[
C_m^k = C_m^k(x, n, r) = \frac{1}{n^m} \sum_{i,j} |d_m(x_i^n, x_j^n)| \leq r \tag{9}
\]
Here, as above, the quantity depends only on the beginning \( x_0^n \) of \( x \). \( C_m^k \) measures the relative frequency of recurrences (in the \( m \)th embedding) followed by at least \( (k - 1) \) other recurrences. Since, in a diagonal line of length \( l \geq k \), just the first \( (l - k + 1) \) points are followed by \( (k - 1) \) other recurrences, it immediately follows that
\[
C_m^k = \frac{1}{n^m} \sum_{l \geq k} (l - k + 1) L_l^n \tag{10}
\]
for every \( m, k \geq 1 \). Comparison with (6) gives the next statement.

Proposition 1. For \( m, k \geq 1 \),
\[
RR^m_n = k \cdot C_m^k(n, k) - (k - 1) \cdot C_m^{k-1}.
\]

Proof. By (10) and (6) we have
\[
\begin{align*}
n^2RR^m_n &= k \left[ n^2C_m^k - \sum_{l \geq k + 1} (l - k + 1) L_l^n \right] \\
&+ \sum_{l \geq k + 1} L_l^n = kn^2C_m^k - (k - 1) \sum_{l \geq k + 1} (l - k) L_l^n \\
&= kn^2C_m^k - (k - 1) n^2C_m^{k-1},
\end{align*}
\]
from which the assertion immediately follows. ■

Validity of the previous relation can be also seen from the following picture
\[
\cdots \ o \cdots \ o \cdots \ o \cdots \ o \cdots \ o \cdots \ o \cdots \ o \cdots \ o
\]
of a diagonal line of length \( l = |a| + |b| \geq k \).
The \( a \)-dots are counted in the \( k \)-recurrence rate as well as in both the \( k \) and \( (k - 1) \)-correlation sum. On the other hand, all of the \( b \)-dots are counted in the \( k \)-recurrence rate, but only the first one is counted in the \( k \)-correlation sum and none in the \((k - 1)\)-correlation sum. This gives \( RR_m = k(C_m^k - C_m^{k-1}) + C_m^{k-1} \), which is equivalent to the formula from Proposition 1.

As a corollary of Proposition 1 we can immediately obtain the formula (3) for the determinism in terms of correlation sums. Since (10) gives
\[
(1/n^2) \sum_{l \geq k} L_l^n = C_m^k - C_m^{k+1},
\]
we also obtain that
\[
LAVG^m_n = k - \frac{C_m^k - C_m^{k+1}}{C_m^k}. \tag{11}
\]

As was noted by many authors, if the metric \( g^n \) in the embedding space is the Chebyshev one (see (4)), the embedded recurrence quantities can be expressed in terms of the nonembedded ones. Let us formulate this as a lemma; there, \( L_q, C_q, RR_q \) stand for \( L_q^k, C_q^k, RR_q^k \), respectively.

Lemma 2. Let \( m, k \geq 1 \) and \( g^n \) be given by (4). Then
\[
L_q^n(x, n, r) = L_q(x, n', r),
\]
where \( n \in \mathbb{Z} \), \( h \in \mathbb{Z} \), and \( m' = n + m - 1 \).

Proof. By (4), \( d^n_k(\mathbf{x}, \mathbf{x}^\infty) \leq r \) if and only if \( \rho(x_{i+1}, x_{j+1}) \leq r \) for every \( 0 \leq i < h \). Thus the first equality is an immediate consequence of the definition of diagonal lines and the second one follows from (10). Further, (6) gives

\[
\begin{align*}
\rho^2 RR_k^{n}(x, n, r) & = \sum_{i \leq k} (|i - (m - 1)|) \cdot L_k(x, n', r) \\
& = (n')^2 \cdot RR_k(x, n', r) - (m - 1) \\
& \quad \sum_{i \leq k} L_k(x, n', r).
\end{align*}
\]

Hence, using (10), also the third formula is proved.

3. Strong Laws for RQA

Here, among other results, we formulate and prove strong laws of large numbers for the recurrence rate and determinism. First, the necessary notions and results are summarized. By a space we always mean a topological space.

3.1. Preliminaries

Let \( Z \) be a space and \( B_Z \) be the Borel \( \sigma \)-algebra on \( Z \). A (measure-theoretical) dynamical system is a quadruple \( (Z, B_Z, \mu, T) \), where \( \mu \) is a probability measure on \( (Z, B_Z) \) and \( T : Z \rightarrow Z \) is a (Borel) measurable map which preserves \( \mu \), that is, \( \mu(T^{-1}(B)) = \mu(B) \) for every \( B \subseteq B_Z \). A set \( B \subseteq B_Z \) is said to be \( T \)-invariant if \( T^{-1}(B) = B \). We say that \( T \) is \( \mu \)-ergodic or that \( \mu \) is \( T \)-ergodic if \( \mu(B) \in \{0,1\} \) for every \( T \)-invariant set \( B \). For \( n \in \mathbb{N} \), the \( n \)th (forward) iterate \( T^n \) of \( T \) is defined recursively by \( T^0 = \text{id}_Z \) and \( T^{n+1} = T \circ T^n \). For \( n \geq 0 \) and \( x \in Z \) we write \( T^n(x) \) instead of \( (T^n)^{\circ n}(x) \). Let \( S \) be a separable metric space, \( B_S \) the Borel \( \sigma \)-algebra on \( S \). On the product space \( S^\infty \), the Borel \( \sigma \)-algebra is denoted by \( B_S^\infty \). An \( S \)-valued (discrete time) stochastic process is a sequence \( X = \{X^n, (n \in \mathbb{N}) \} \) defined on a probability space \( (\Omega, \mathcal{B}, \mu) \). The distribution of the process \( X \) is the measure \( \mu \) on \( (S^\infty, B_S^\infty) \) defined by \( \mu(F) = \mathbb{P}(X^\infty \in F) \).

Theorem 3. Let \( S \) be a separable metric space, \( X \) be an \( S \)-valued ergodic stationary process with distribution \( \mu \) and \( m, k \geq 1 \) be integers. Then for

\[
C_k^{n}(x, n, r) = \left( \frac{n'}{n} \right)^2 \cdot C_k(x, n', r)
\]

and

\[
RR_k^{n}(x, n, r) = \left( \frac{n'}{n} \right)^2 \cdot RR_k(x, n', r) - (m - 1) \cdot (C_k(x, n', r) - C_{k+1}(x, n', r)).
\]

The (left) shift on \( S^\infty \) is the (continuous) map \( T : S^\infty \rightarrow S^\infty \) defined by

\[
T(x^n) = y^n, \quad \text{where } y_n = x_{n+1} \text{ for every } n \in \mathbb{N}.
\]

Let \( \pi : S^\infty \rightarrow S \) denote the projection onto the zeroth coordinate, that is, \( \pi(x^n) = x_0 \). If \( X \) is a stochastic process with distribution \( \mu \), then shift \( T \) together with the projection \( \pi \) and the measure \( \mu \) form the Kolmogorov representation of the process \( X \). From now on, we always assume that \( X \) is directly given by its Kolmogorov representation, that is,

\[
(\Omega, \mathcal{B}, \mu) = (S^\infty, B_S^\infty, \mu) \quad \text{and} \quad X_n = \pi \circ T^n.
\]

A process \( X \) is (strictly) stationary if its distribution \( \mu \) is \( T \)-invariant. The marginal of a stationary process \( X^\infty \) is the distribution of \( X_0 \). A process \( X \) is ergodic if every \( T \)-invariant set has probability either 0 or 1. Thus, a process \( X \) is stationary and ergodic if and only if the dynamical system \( (S^\infty, B_S^\infty, \mu, T) \) is ergodic.

3.2. Strong law for correlation sum

For a Borel measure \( \mu \) on \( S^\infty \), \( m, k \geq 1 \) and \( r \geq 0 \) define the correlation integral \( c_k^{n}(r) \) by

\[
c_k^{n}(r) = \mu(\{x, y : d^n_k(x, y) \leq r\}).
\]

If \( \mu \) is the distribution of a process \( X^\infty \), then \( c_k^{n}(r) \) is the probability that, for two independent random vectors \( Y_i \ldots Y_{i+k-1} \) with the distribution equal to that of \( X_0 \ldots X_{i+k-1} \), every \( Y_i \ldots Z_{i+k-1} \) are \( r \)-close according to \( \rho^2 \):

\[
c_k^{n}(r) = \mu(\rho^2(Y_i \ldots Z_{i+k-1}) \leq r \text{ for every } 0 \leq i < k).
\]

The following theorem, the proof of which is postponed to Sec. 7, follows from [Manning & Simon, 1998].

Theorem 3. Let \( S \) be a separable metric space, \( X \) be an \( S \)-valued ergodic stationary process with distribution \( \mu \) and \( m, k \geq 1 \) be integers. Then for
Theorem 5 (Strong laws for \( \text{DET} \) and \( \text{LAVG} \)). Under the assumptions of Theorem 3, for \( \mu \)-a.e. \( x \in S^\infty \) and for every (up to countably many) \( r > 0 \),

\[
\lim_{n \to \infty} \text{DET}_n^m(x, n, r) = \text{DET}^m(x, r)
\]

and

\[
\lim_{n \to \infty} \text{LAVG}_n^m(x, n, r) = \text{LAVG}^m(x, r).
\]

Proof. The statements follow since a.e.-convergence is preserved by elementary arithmetic operations provided that, for division, the numerator or denominator is nonzero. \( \blacksquare \)

Remark 6. Theorem 3 can be trivially used to derive strong law also for another RQA quantity called the \( k \)-ratio defined by

\[
\text{RATIO}^m_k = \frac{\text{DET}^m_k}{\text{RR}^m_1}.
\]

Further, under the assumptions of Theorem 3, the maximal diagonal line length \( \text{LMAX}^m \) defined by

\[
\text{LMAX}^m(x, n, r) = \max\{l < n : L^m_l(x, n, r) > 0\}
\]

diverges (for \( \mu \)-a.e. \( x \in S^\infty \) and for every \( r > 0 \)) to infinity; this follows from the Birkhoff ergodic theorem. As a corollary we immediately have that the reciprocal value \( \text{DIV}^m = 1/\text{LMAX}^m \) called the divergence converges almost surely to zero.

Remark 7. Recurrence measures as well as correlation sums are often defined using strict inequalities \( d^m_k(x, y) < r \), and/or with excluding the main diagonal \( i = j \). Clearly, the latter has no effect on asymptotic properties, that is, Theorems 3-5 remain true also in this case. When one uses strict inequalities, then again the results are valid provided the strict inequality is used also in the definition (12) of the correlation integral. The relationship between this new “open” correlation integral and the “closed” one is straightforward, see [Pesin & Tempelman, 1995, Remark 2.2].

Remark 8. As can be seen from Theorem 19, Theorem 3 is valid with \( d^m_k \) replaced by any separable Borel pseudometric \( d \) on \( S^\infty \). For example, \( d \) can be defined via order patterns (cf. [Amigó, 2010]):

\[
d(x^m_k, y^m_k) = 1 \text{ if } x^m_k, y^m_k \text{ have the same order pattern, } d(x^m_k, y^m_k) = 0 \text{ otherwise.}
\]

In this way, we obtain strong laws for RQA characteristics based on order patterns recurrence plots.

As for “empirical” RQA quantities (see Lemma 2), the dependence of asymptotic ones on the embedding dimension \( m \) is straightforward provided the maximum metric is used.

Lemma 9. Let \( m, k \geq 1 \), \( r \geq 0 \) and \( \rho^m \) be given by (4). Then

\[
\text{c}_k^m = c_h, \quad \text{RR}^m_k = \text{RR}_k - (m - 1)(c_h - c_{h+1}) \quad \text{and} \quad \text{LAVG}^m_k = \text{LAVG}_h - (m - 1),
\]

where \( h = k + m - 1 \).

Proof. The first equality follows from (12) and the definition (5) of \( d^m_k \). The others are then consequences of (15). \( \blacksquare \)
3.4. Asymptotic determinism via conditional probabilities

Here we assume (4). For $h, l \geq 1$ and $r > 0$ define the conditional correlation integral by

\[ c_{hl}(r) = \frac{c_{hl}(r)}{c_{l}(r)} \]

\[ = \frac{\mu(\{ (y,z) : d_{hl}(y,z) \leq r \})}{\mu(\{ (y,z) : d_{l}(y,z) \leq r \})}. \]

(16)

Particularly, if $\mu$ is the distribution of an ergodic stationary process $S$, then $c_{hl}(r)$ is the conditional probability

\[ c_{hl}(r) = \mu(\{ (y,z) : d_{hl}(y,z) \leq r \}) \]

\[ = \mu(\{ (y,z) : d_{l}(y,z) \leq r \}). \]

Thus, $c_{hl}(r)$ is the probability that $h$ consecutive recurrences are followed by at least $l$ others. In view of this we have the following interesting expression of asymptotic determinism in terms of conditional probabilities.

\[ \text{Theorem 10. Under (4), the asymptotic determinism can be expressed via a linear combination of conditional correlation integrals} \]

\[ \det^m_{hl} = k \cdot c_{hl} - (k-1) \cdot c_{l}. \]

Consider now the special case of (ergodic stationary) Markov processes of order $p \geq 1$. Then for every $m \geq p$ one has $c_{lm} = c_{lp}$. That is, over-embedding has no effect on the asymptotic determinism.

\[ \text{Corollary 11. For every (ergodic stationary) Markov process of order $p \geq 1$ and for every $m \geq p$,} \]

\[ \det^m_{ll} = \det^m_{l}. \]

4. Asymptotic RQA Measures for Some Processes

Now we present some applications of the asymptotic results obtained in the previous section. We assume that $S = (S, \rho)$ is a separable metric space and $S_i$ is equipped with the pseudometric $d_{li}$ given by (5), where $m$ is the embedding dimension, $k$ is the prediction horizon and the embedding metric $\rho_i$ is given by (4). We also assume that $X^n_0$ is a Kolmogorov representation of an ergodic stationary $S$-valued process. In the following we derive explicit formulas for asymptotic RQA measures for some classes of processes. To make the paper self-contained we include here also the proofs, though the results (at least for correlation integrals) are known. The convergence is demonstrated by simulation studies. We start with the simplest case of iid processes.

4.1. IID processes

**Proposition 12.** Let $X^n_0$ be an iid process. Then, for $m, k \geq 1$ and $r > 0$,

\[ c_{kl}(r) = \alpha^{m+k-1}, \text{ where } \alpha = c(r). \]

Hence

\[ \det^m_{kl}(r) = \alpha^{k-1} |k - (k-1)\alpha| \]

\[ \text{and } \text{lavg}^m_{kl}(r) = k + \frac{\alpha}{1 - \alpha}, \]

do not depend on the embedding dimension $m$.

**Proof.** By Lemma 9 we may assume that $m = 1$. Let $Y^n_0, Z^n_0$ be independent random vectors with the distribution equal to that of $X^n_0$. Then, for every $r > 0$,

\[ c_{k}(r) = \mu(\{ (Y^n_0, Z^n_0) : d_{k}(y,z) \leq r \}) \]

\[ = \mu(\{ (Y^n_i, Z^n_i) : d_{k}(y,z) \leq r \} \text{ for every } 0 \leq i < k \}) \]

\[ = \prod_{0 \leq i < k} \mu(\{ (Y^n_i, Z^n_i) : d_{k}(y,z) \leq r \}) \]

\[ = \alpha^{k-1} |k - (k-1)\alpha| \]

Thus the first statement is proved. The rest follows from the definitions (15) of $\det^m_{kl}$ and $\text{lavg}^m_{kl}$. \( \blacksquare \)

For example, the asymptotic determinism of a Gaussian iid process with the variance $\sigma^2$ is

\[ \det^m_{kl}(r) = 2k\Phi(r') - 1]^{k-1} \]

\[ \left[ 1 - \frac{1}{2} \right]^k - \Phi(r') \right], \text{ where } r' = \frac{r}{\sqrt{2\sigma}}, \]

and $\Phi$ is the distribution function of the standard normal distribution. For this, use the fact that for iid Gaussian random variables $Y, Z$ with the variance $\sigma^2$, $Y - Z \sim N(0, 2\sigma^2)$ and so $c(r) = \mu(\{ (Y - Z) : d_{k}(y,z) \leq r \}) = 2\Phi(r') - 1$.

Figure 1 illustrates the convergence of the empirical determinism (with $m = 1$ and $k = 2$) to the asymptotic one for a Gaussian iid process.

4.2. Markov chains

Let $S = \{0, 1, \ldots, q-1\}$ be a finite space equipped with the discrete metric $\rho$ (that is, $\rho(x,y) = 1$
∑ quantities do not depend on \( r \) setting, only the distance threshold embedding dimension. Notice that, in this discrete mean diagonal line length do not depend on the also here we can see that both the determinism and given in the following proposition. As in the iid case, also here we can see that both the determinism and mean diagonal line length do not depend on the embedding dimension. Notice that, in this discrete setting, only the distance threshold \( r \) less than 1 needs to be considered and, for \( 0 \leq r < 1 \), RQA quantities do not depend on \( r \).

**Proposition 13.** Let \( X^\infty_0 \) be a finite-valued Markov chain with the transition matrix \( P \) and the stationary distribution \( \pi \). Then, for \( r \in [0, 1) \),

\[
\begin{align*}
\det^m_0(r) &= \alpha \beta^k + m - 2, \\
\text{cav}^m_0(r) &= \beta^{k-1}[k - (k - 1)] \beta^{-1} \text{ and } \frac{\beta \text{det}^m_0(r)}{\alpha} = k - \frac{\beta}{1 - \beta}
\end{align*}
\]

where \( \alpha = c(r) = \pi' \pi \) and \( \beta = c_{\mu}(r) = \pi' \text{diag}(PP') \pi / \alpha \).

**Proof.** Only the equality for \( \text{cav}^m_0(r) \) needs a proof since the other two follow from (15). We may assume that \( m = 1 \). Let \( Y^k_0, Z^k_0 \) be independent random vectors with the distribution equal to that of \( X^k_0 \). Fix \( r < 1 \); then \( \alpha = \mu(D[Y_0, Z_0]) \leq r \) if and only if \( \mu(D[Y_0, Z_0] \leq r) \) and \( \beta = \mu(D[Y_0, Z_0]) \leq r \). If \( k = 1 \) we are done. So assume that \( k \geq 2 \). Then

\[
\beta = \frac{1}{\alpha} \mu(D[Y_1, Z_1]) \leq r
\]

\[
= \frac{1}{\alpha} \sum_{s,t} \mu(Y_0 = Z_0 = s, Y_1 = Z_1 = t)
\]

\[
= \frac{1}{\alpha} \sum_{s,t} \mu(X_0 = s, X_1 = t)^2
\]

\[
= \frac{1}{\alpha} \sum_{s,t} \pi_s \pi_t \rho_{at}^2 = \frac{1}{\pi} (\pi' \text{diag}(PP') \pi).
\]

Since \( X^k_0 \) is a stationary Markov chain, we obtain

\[
\begin{align*}
c_k(r) &= \mu(D[Y_k, Z_k]) \leq r \\
&= \mu(Y_i = Z_i \forall 0 \leq i < k) \\
&= \mu(Y_{k-1} = Z_{k-1} | Y_i = Z_i \forall 0 \leq i < k - 1) \\
&= \beta c_{k-1}(r) \\
&= \beta c_{k-1}(r).
\end{align*}
\]

Now a simple induction gives the desired result. \( \blacksquare \)

Figure 2 depicts the convergence of the empirical determinism (with \( m = 1 \)) to the asymptotic one for a 3-state Markov chain with the (randomly selected) transition matrix

\[
P = \begin{pmatrix}
0.362 & 0.438 & 0.200 \\
0.484 & 0.447 & 0.069 \\
0.120 & 0.503 & 0.377
\end{pmatrix}.
\]
4.3. Autoregressive processes

Next we consider asymptotic RQA characteristics of a (stationary) autoregressive process $X_n^\infty \sim AR(p)$ of order $p \geq 1$ with coefficients $\theta_i$ ($i = 1, \ldots, p$) and with the Gaussian zero mean noise $\epsilon_n$ of variance $\sigma^2$. It is given by $X_n = \theta_1 X_{n-1} + \theta_2 X_{n-2} + \cdots + \theta_p X_{n-p} + \epsilon_n$.

Proposition 14. Let $X_n^\infty$ be an (ergodic stationary) autoregressive process $AR(p)$ with coefficients $\theta_1, \ldots, \theta_p$ and Gaussian $WN(0, \sigma^2)$. Let $r > 0$, $m, k \geq 1$ and $h = k + m - 1$. Then

$$c_m^k(r) = \mu\{Y_0^h \in [-r, r]^h\},$$

where $Y_0^h \sim N(0, \Sigma)$ with $\Sigma$ being the $h \times h$ autocovariance matrix of an $AR(p)$ process with coefficients $\theta_1, \ldots, \theta_p$ and Gaussian $WN(0, 2\sigma^2)$.

Proof. Since the difference of two independent $AR$ processes with the same parameters $\theta_1, \ldots, \theta_p, \sigma^2$ is an $AR(p)$ process with the same coefficients and noise variance $2\sigma^2$, the statement immediately follows from (13). $\blacksquare$

The convergence of the empirical determinism $DETR$ to the asymptotic one for an AR process is illustrated in Fig. 3.

Table 2. Determinisms for the $AR(3)$ process with $a_1 = 0.25$, $a_2 = 0.4$, $a_3 = 0.3$ and $\sigma^2 = 1.5$; here $n = 2500$ and $r = \sqrt{1.5}$. Average determinisms with standard errors in parentheses obtained by a Monte Carlo simulation of size 1000.

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.638</td>
<td>0.725</td>
<td>0.759</td>
<td>0.760</td>
<td>0.760</td>
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<tr>
<td></td>
<td>(0.009)</td>
<td>(0.009)</td>
<td>(0.009)</td>
<td>(0.009)</td>
<td>(0.009)</td>
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<tr>
<td>3</td>
<td>0.407</td>
<td>0.491</td>
<td>0.514</td>
<td>0.515</td>
<td>0.516</td>
</tr>
<tr>
<td></td>
<td>(0.010)</td>
<td>(0.011)</td>
<td>(0.012)</td>
<td>(0.013)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>4</td>
<td>0.258</td>
<td>0.312</td>
<td>0.327</td>
<td>0.328</td>
<td>0.329</td>
</tr>
<tr>
<td></td>
<td>(0.010)</td>
<td>(0.011)</td>
<td>(0.012)</td>
<td>(0.013)</td>
<td>(0.014)</td>
</tr>
<tr>
<td>5</td>
<td>0.158</td>
<td>0.191</td>
<td>0.201</td>
<td>0.201</td>
<td>0.202</td>
</tr>
<tr>
<td></td>
<td>(0.008)</td>
<td>(0.010)</td>
<td>(0.011)</td>
<td>(0.012)</td>
<td>(0.013)</td>
</tr>
</tbody>
</table>
Corollary 11 implies that over-embedding of an $AR(p)$ process into dimension $m > p$ leaves the determinism unchanged. This can be used to estimate (from below) the order of an autoregressive process, which is demonstrated in Table 2 for an $AR(3)$ process. There one can see that embedding into dimension 4 or 5 gives the determinism equal to that for dimension 3, but the determinisms for $m = 1, 2$ are smaller. Thus one can conclude that the order of the process is at least 3.

5. Metric Entropy and Asymptotic Determinism

In the following three examples we demonstrate that the behavior of the RQA determinism can sometimes be counterintuitive. First, we show that the determinism of an iid process can be higher than that of a non-iid one with the same marginal. In the second example it is shown that the higher metric entropy does not necessarily mean smaller determinism. Finally, a Markov chain indistinguishable (from the RQA point of view) from an iid process is constructed.

Example 15 [Determinism of iid and non-iid processes]. Fix $0 < a, b < 1$ and consider a 01-valued Markov chain $X_n^\alpha$ with the transition matrix $P = (p_{ij})_{i,j=0}^1$ such that $p_00 = a, p_{11} = b$. Then $X_n^\alpha$ is ergodic and the stationary distribution of it is given by $\pi = (\pi_0, \pi_1) = \left(\frac{1}{a-b}, \frac{b}{a-b}\right)$. Fix any $r \in (0, 1)$. By Proposition 13,

$$\det_0^\alpha(r)\text{Markov} = k\beta^{k-1} - (k-1)\beta^k,$$

where

$$\beta = c_{11}(r) = \frac{(1 - a)^2 \cdot [b^2 + (1-b)^2] + (1-b)^2 \cdot [a^2 + (1-a)^2]}{(1-a)^2 + (1-b)^2},$$

On the other hand, for a 01-valued iid process with the same marginal $\pi$, Proposition 12 gives

$$\det_0^\pi(r)\text{iid} = k\alpha^{k-1} - (k-1)\alpha^k,$$

where

$$\alpha = \frac{(1 - a)^2 + (1-b)^2}{(2 - a - b)^2}.$$

If we take $a = 3/5$ and $b = 1/5$, then $\alpha > \beta$. Since the function $x \mapsto kx^{k-1} - (k-1)x^k$ is increasing on $[0, 1)$, we have that $\det_0^\pi(r)\text{iid} < \det_0^\pi(r)\text{Markov}$ for any $m, k \geq 1$.

Example 16 [Determinism and metric entropy]. The previous example also shows that the higher metric entropy does not necessarily mean smaller determinism. In fact, the metric entropy of an iid process is strictly greater than that of any stationary non-iid process with the same marginal. In this simple case the entropies can be calculated analytically, since the metric entropy of an iid process is $h_{\pi}^\text{iid} = -\sum \pi_a \log \pi_a$ and the metric entropy of the Markov chain is $h_{\pi}^\text{Markov} = -\sum \pi_a \log \pi_a$. See also Fig. 4 for an illustration of this phenomenon.

Example 17 [Indistinguishable Markov chain and iid process]. In the 2-state Markov chain considered in Example 15, fix $b = 1/5$ and, for given $a$, denote by $\alpha_a, \beta_a$ the corresponding correlation integrals $c_{11}(r), c_{11}(r)$, respectively. Since $\alpha_{1/2} < \beta_{1/2}$ and $\alpha_{3/5} > \beta_{3/5}$, there is $a \in (1/2, 3/5)$ with $\alpha_a = \beta_a$. For this particular (non-iid) Markov chain $X_n^\alpha$, the probability of finding a diagonal line of length $k$ (in the infinite recurrence plot) is the same as that for an iid process $Y_n^\alpha$ with the same marginal. Hence, no RQA measure based on diagonal lines can distinguish between $X_n^\alpha, Y_n^\alpha$.

6. Spurious Structures

In [Thiel et al., 2006], see also [Thiel et al., 2002] and [Marwan et al., 2007, Section 3.2.4], it was pointed out that, for iid processes, over-embedding leads to the existence of spurious structures in recurrence plots. The appearance of spurious structures is illustrated in Fig. 5. The left panel depicts the “usual” recurrence plot of an iid process for the embedding dimension $m = 1$. On the right panel,
there is the recurrence plot for the embedding dimension $m = 250$. It contains long diagonal lines, which would suggest that the process should be well predictable.

Proposition 12 enables us to explain why this happens. In fact, this is due to a special choice of the distance threshold $r$, which selects $r = r_m$ such that the recurrence rate $RR_m^\theta(r_m)$ is fixed to a predetermined level. As the following proposition demonstrates, this selection rule leads to the determinism close to one and average diagonal line arbitrarily high for large embedding dimensions.

Proposition 18. Let $X^\infty_\theta$ be an iid process. Let $\theta > 0$ and let $r_m > 0$ ($m \in \mathbb{N}$) be such that all the recurrence rates $RR_m^\theta(r_m)$ are equal to $\theta$. Then, for $k \geq 1$,

$$\lim_{m \to \infty} \det_k^\theta(r_m) = 1 \quad \text{and} \quad \lim_{m \to \infty} \operatorname{avg}^\theta_l(r_m) = \infty.$$

Proof. For $m \geq 1$ put $\alpha_m = c(r_m)$. Then, by the assumption and Proposition 12, $\theta = RR_m^\theta(r_m) = (\alpha_m)^m$ for every $m$; thus $\alpha_m = \theta^1/m \to 1$ for $m \to \infty$. Using Proposition 12 we obtain that, for every $k \geq 1$, $RR_k^\theta(r_m) = (\alpha_m)^m(k+1)$, $[k - (k - 1)r_m] \to \theta$ for $m \to \infty$, and so $\lim_{m} \det_k^\theta(r_m) = 1$ and $\lim_{m} \operatorname{avg}^\theta_l(r_m) = \infty$. \hfill \blacksquare

Hence, appearance of the spurious structures for iid processes is an artefact of this particular selection rule for density thresholds. The artificial “predictability” which appears on the right panel of Fig. 5 is due to the distance threshold $r$, which is several times higher than the standard deviation of the process. Different selection rule, which chooses $r$ independently of the embedding dimension, leaves the determinisms $\det_k^\theta(r)$ and mean diagonal line lengths $\operatorname{avg}^\theta_l(r)$ constant for $m \to \infty$, as one expects for iid processes.

7. Strong Law for Correlation Sums on Pseudometric Spaces

Here we give a proof of Theorem 3, based on the strong law for correlation sums on pseudometric spaces; see Theorem 19 below. Recall that, for a (topological) space $Z$, a map $d : Z \times Z \to \mathbb{R}^+$ is a pseudometric on $Z$ if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in Z$. A pseudometric $d$ is separable if the topology generated by it is separable. We say that $d$ is a Borel (continuous) pseudometric on $Z$ if it is a pseudometric which is Borel (continuous) w.r.t. the product topology on $Z \times Z$. Notice that a continuous pseudometric on a separable space is automatically separable.

If $d$ is a pseudometric on $Z$, $B_d(x, r)$ and $S_d(x, r)$ denote the (closed) $d$-ball and $d$-sphere with radius $r$ centered at $x$, respectively. Notice that if $d$ is Borel then $d$-balls and $d$-spheres are analytic sets in $Z$; to see it, use that $B_d(x, r) = \{y : (x, y) \in d^{-1}(0, r]\}$ and analogously for $S_d(x, r)$. The $d$-diameter of a set $A \subseteq Z$ is denoted by $\operatorname{diam}_d(A)$.
Assume that \((Z, \mathcal{B}_Z, \mu, T)\) is a dynamical system and that \(d\) is a Borel pseudometric on \(Z\). For \(x \in Z, n \in \mathbb{N}\) and \(r \geq 0\) define the correlation sum

\[
C_d(x, n, r) = \sum_{i,j} \mathbb{1}\{d(T^i(x), T^j(x)) \leq r\}
\]

and the correlation integral

\[
c_d(r) = \mu \{ (x, y) : d(x, y) \leq r \}
\]

(In the integral, the completion of \(\mu\) is used.) Recall that \(c_d\) is nondecreasing, right continuous and tends to 1 if \(r \to \infty\). Further, \(c_d\) is continuous at \(r\) if and only if \(\mu S_d(x, r) = 0\) for \(\mu\)-a.e. \(x \in Z\), see e.g. [Pesin & Tempelman, 1995, Remark 2.2].

The strong law for the correlation sum was studied under different conditions in [Pesin, 1993; Pesin & Tempelman, 1995; Aaronson et al., 1996; Serinko, 1996; Manning & Simon, 1998]. Though not stated in this form, the following theorem was proved in [Manning & Simon, 1998].

**Theorem 19.** Let \(Z\) be a topological space, \(\mu\) be a Borel probability on \(Z\) and \(T: Z \to Z\) be a \(\mu\)-ergodic Borel map. Let \(d\) be a separable Borel pseudometric on \(Z\). Then, for \(\mu\)-a.e. \(x \in Z\) and for every \(r > 0\),

\[
\lim_{n \to \infty} C_d(x, n, r) = c_d(r) > 0
\]

provided \(c_d\) is continuous at \(r\).

Let us note that this “pseudometric” version of the strong law for correlation sums cannot be directly derived from the “metric” one. Indeed, it is true that one can easily obtain a metric space from the pseudometric one by gluing together points of zero distance, as is usually done. The considered dynamical system, however, does not necessarily fit to this projection and so, in general, there is no induced system on the obtained metric space; take e.g. the case when \(Z = \mathbb{R}^\infty\), \(T\) is the shift and \(d(x, y) = |x - y|\).

The proof from Manning and Simon [1998], however, perfectly fits to this general setting, as was noted by the authors. Indeed, it is based on the Birkhoff ergodic theorem and on the existence of finite Borel partitions \(\mathcal{A}_n = \{ A_j^n : 0 \leq j \leq M_n \}\) \((m \geq 1)\) with \(\mu(A_j^n) \leq 2^{-m}\) and \(\text{diam}_d(A_j^n) \leq 2^{-m}\) for every \(j \geq 1\). Since the former is true for arbitrary ergodic system and the latter immediately follows from separability of \((Z, d)\) and analyticity of \(d\)-balls, the convergence in Theorem 19 can be proved using the same reasoning as in [Manning & Simon, 1998]. Finally, the fact that \(c_d(r) > 0\) for every \(r > 0\) is obvious due to separability of \((Z, d)\). (For this, take any Borel set \(B\) with \(d\)-diameter at most \(r\) and \(\mu(B) > 0\) and use \(c_d(r) \geq \mu(B)^2\).)

Next we show how Theorem 3 can be obtained from Theorem 19.

**Proof.** Let \((S, \rho)\) be a separable metric space, \(m, k \geq 1\) be integers and \(r > 0\) be such that \(C_m^\infty\) is continuous at \(r\). Let \(g_m\) be a metric on \(S^m\) compatible with the product topology. Take \(Z = S^\infty\) and define \(d = d_m^\infty\) by (5). Then obviously \(d\) is a continuous pseudometric on \(Z\); it is separable due to separability of \(Z\).

Let \(X_m^\infty\) be an \(S\)-valued ergodic stationary process with distribution \(\mu\). We may assume that \(X_m^\infty\) is given by its Kolmogorov representation, that is, \(X_n = \sigma \circ T^n\), where \(T: Z \to Z\) is the shift and \(\sigma: Z \to S\) is the projection \(x^\infty \to x_0\). Then, for every \(x = x_0^\infty \in Z\) and every \(n, T^n(x) = x_n\) and so

\[
c_m^\infty(r) = c_d(r) \quad \text{and} \quad C_m^\infty(x, n, r) = \left( \frac{n-k+1}{n} \right)^2 C_d(x, n-k+1, r).
\]

Application of Theorem 19 to the ergodic system \((Z, \mathcal{B}_Z, \mu, T)\) gives the desired result. ■

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