An examination of nonlinear dependence in exchange rates, using recent methods from chaos theory

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Abstract

Interest in the relevance of nonlinear dynamics to finance and economics has spurred the evolution of new ways to analyze time series data. Tests for chaos, based on a metric approach which measures spatial correlations, led to the development of the correlation dimension test for chaos and the BDS test for nonlinearity. More recently, a topological method has been introduced into the scientific literature which employs a simple qualitative test for chaos that is adaptable to the characteristics of financial data. A quantitative version is also presented here. Conflicting evidence exists about the presence of chaotic behavior in exchange-rate data. The qualitative topological test does not support evidence of a chaotic generating mechanism in these series. The quantitative form finds nonlinear dependence and is a useful diagnostic to determine the adequacy of ARCH-type models for this nonlinear structure.

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1. Introduction

The characterization and modeling of exchange-rate behavior is an issue which has generated an extensive body of research. The view was long prevalent that the logarithmic first differences of exchange rates were statistically independent and that exchange rates thus followed a random walk (Mussa, 1979). Examination of autocorrelations found little evidence of linear dependence, and numerous models were developed based on that assumption. However, more recent research produced evidence of nonlinear dependence. One explanation

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of the presence of nonlinear dependence is that exchange rate changes are nonlinear stochastic functions of their own past values. This led to the use of autoregressive conditionally heteroscedastic models of exchange rates, e.g., Baillie and Bollerslev (1989), Bollerslev (1987), Brock, Hsieh, and LeBaron (1991), De Grauwe et al. (1993), Fernandes (1998), Hsieh (1989a, 1989b), Malliaropulos (1997), McKenzie (1997), Peel and Spreight (1994), and Theodossiou (1994). A second possible explanation is that the series are generated by deterministic chaotic processes. Using metric methods to test for chaos, De Grauwe et al. (1993) reported indications of possible chaos in several exchange rate series. Brock et al. (1991) and Hsieh (1989b) found evidence of nonlinear dependence in several exchange rate series, but their results failed to support a chaotic interpretation.

The present research explains and utilizes newer methods to analyze nonlinearity derived from a topological approach to chaos. These methods are applied to daily exchange-rate data to shed additional light on the nature of the nonlinearity present in exchange rates.

Metric tests for chaos are explained in Section 2. In Section 3 on methodology the topological approach to chaos is discussed and applied to analyzing the structure of exchange rate series. Conclusions and directions for future research are contained in Section 4.

2. Review of literature on tests for chaos

Chaos, which may appear to be random but is generated by a deterministic model, cannot be detected by standard statistical techniques, such as autocorrelation functions (Sakai & Tokumaru, 1980). In 1983 a metric procedure was developed in the physics literature by Grassberger and Procaccia (1983) to identify chaotic behavior in time series data. It is based on estimating a correlation dimension, which is a measure of the relative rate of scaling of the density of points in a given space. The underlying concept is straightforward. Dimension measures the number of degrees of freedom relevant to a system’s dynamical behavior. The true, unknown system that generates the time series \( \{x_t\} \) is \( m \)-dimensional. The \( \{x_t\} \) can be transformed into a sequence of \( m \)-tuples \((x_{t\tau}, x_{t+\tau}, \ldots x_{t+(m-1)\tau})\), where \( \tau \) is a time-delay parameter. Each \( m \)-tuple represents a point in an \( m \)-dimensional Euclidean space. This mapping of a time series into an \( m \)-dimensional space is called an \((m\)-dimensional\) embedding of the data.

The \( m \)-tuples can be plotted and the properties of the resulting set of points can be studied. If the time series results from a random process, then as the embedding dimension is increased, the \( m \)-tuples fill the space in each direction; the process is infinite-dimensional. But if the time series is generated by a deterministic model, the time series path will be attracted to a subset of points in the space, known as an attractor. As the embedding dimension \( m \) increases, the dimension of the points will not increase beyond some limit, \( k \), and the dimension will be less than \( m \). The details of the correlation dimension test are described elsewhere in the economics and finance literature, including Barnett and Chen (1988), Brock (1986), Brock et al. (1991), and Chyi (1997).

While analysis of economic and financial time series data using the correlation dimension test initially produced some evidence consistent with chaos (Barnett & Chen,
1988; Frank & Stengos, 1989; Sayers, 1987), the limitations of this test when applied to financial data were soon realized. Specifically, the correlation dimension test did not work well on relatively short data sets (several thousand observations, or less), noisy data, or nonstationary series. These problems led to development of the BDS test (Brock, Dechert, & Scheinkman, 1987; Dechert, Scheinkman, & LeBaron, 1996), which is derived from the Grassberger–Procaccia method. The null hypothesis for the BDS test, however, is not that the time series is chaotic but rather that it is independent and identically distributed (iid). The alternative hypothesis includes chaos, as well as other types of linear or nonlinear structure. Application of the BDS statistic has produced evidence of nonlinearity in economic and financial data (Brock & Sayers, 1988; Fernandes, 1998; Frank & Stengos, 1989; Hsieh, 1989b; Hsieh, 1991; Scheinkman & LeBaron, 1989). Recent contributions to the application of chaos theory to finance have been made by Mandelbrot (1997, 1999), but these have not yet evolved into a new testing procedure.

3. Methodology

3.1. The close returns test

The present research utilizes a methodology which is distinct from the BDS test, one which is based on a topological approach to chaos (Gilmore, 1998, Mindlin et al., 1990; Tufillaro, 1990; Tufillaro et al., 1990) and applies it to daily exchange-rate data. Referred to as the method of close returns, this procedure has produced both a qualitative and a quantitative test. The former test is designed to detect chaotic structure, while the latter, like the BDS test, is a quantitative method to detect departures from iid. The quantitative test detects both linear and nonlinear structure in a time series, and it can also be applied to the residuals of a model to examine its adequacy. While this approach does not establish a definitive means to distinguish between chaotic and other forms of nonlinear behavior, it is a computationally simple way to provide evidence concerning chaotic or other nonlinear behavior that can be assessed in conjunction with the results of other methods of analysis.

Unlike the correlation-dimension-based methods, the topological approach to chaos searches for a more fundamental characteristic of chaos — the recursive behavior of a chaotic time series, which over time will nearly, although never exactly, repeat itself. Every chaotic attractor contains a number of unstable periodic orbits. The trajectory of a time series evolves near a given unstable periodic orbit for a time, cycling it before being repelled to another area of the attractor. It is this cycling behavior which the close returns test is designed to detect.

The procedure is applied to the time series \( \{x_t\} \) without an embedding. If the observations \( x_t \) evolve near an unstable periodic orbit for a sufficiently long time, they will return to the neighborhood of \( x_t \) after some interval \( l \), where \( l \) indicates the length of the orbit, measured in units of the sampling rate. This means that \( |x_t - x_{t+l}| \) will be small. Further, \( x_{t+1} \) will be near

1 This test is more fully described in Gilmore (1993).
$x_{t+1} = M x_t + N$, $x_{t+2}$ will be near $x_{t+2} + N$, and so on. Thus, it makes sense to look for a series of consecutive data elements for which $|x_t - x_{t+1}|$ is small.

To identify regions of close returns in time series data all differences $|x_t - x_{t+1}|$ are computed. A threshold value, $\varepsilon$, is determined as a small percentage (usually 2% to 5%) of the largest difference between any two values across the data set. This information is presented as a graph. If $|x_t - x_{t+1}| < \varepsilon$, the result is coded black; if $|x_t - x_{t+1}| \geq \varepsilon$, the result is coded white. The horizontal axis of the graph indicates the observation number, $t$, where $t=(1,2,\ldots,N)$, and the vertical axis is $i$, where $i=(1,2,\ldots,N-1)$, and $N$ is the number of observations in the series. Close returns in the data set are indicated by horizontal line segments. If the data set is chaotic, a number of such line segments will be seen in the plot. A chaotic close returns plot (500 by 300 observations) is illustrated in Fig. 1, using data generated from a chaotic model, the Hénon map (Hénon, 1976), whose equations are [Eq. (1)]

$$dx_{t+1} = 1 - ax_t^2 + y_t,$$
$$dy_{t+1} = bx_t. \tag{1}$$

In contrast, a stochastic data set will produce a generally uniform array of black dots. This is illustrated in Fig. 2, using a 500-by-300-observation plot from a simulated pseudorandom time series.

To develop the qualitative close returns test into a quantitative form we construct a histogram from the plot, which records the incidence of close returns “hits.” For each value of $i$ on the vertical axis of the plot the number of close returns is summed across the row, with
Fig. 2. Close returns plot of pseudorandom series. This figure presents a 500-by-300-observation plot of data from a pseudorandom series.

\[ H_i = \Theta(\varepsilon - |x_t - x_{t+1}|) \]  
Here \( \Theta \) is the Heaviside theta function: \( \Theta(x) = +1 \) if \( x \geq 0 \), \( \Theta(x) = 0 \) if \( x < 0 \). The histogram displays \( i \) along the horizontal axis and \( H_i \) along the vertical axis. For the chaotic data the histogram will contain a series of peaks; the histogram for the Henon map is shown in Fig. 3. In contrast, a histogram for the pseudorandom time series constructed from the close returns plot in Fig. 2 will exhibit a scattering around a uniform distribution (Fig. 4).

Fig. 3. Close returns histogram of Henon map. This figure constructs a histogram from the close returns plot, recording the incidence of close returns “hits” for each value of \( i \).
A $\chi^2$ test is now applied to the histogram to determine whether the null hypothesis of iid for the series can be rejected. If the series is iid, then $H_i = \tilde{H}$, a constant. This is tested by calculating the sample $\chi^2_c$. If the data are iid, the distribution described in the close returns plot will be binomial, with the probability of a hit, or $p(\text{black})$, determined by Eq. (2)

$$p(\text{black}) = \frac{\text{total number of close returns}}{\text{total area of plot}}. \quad (2)$$

The average value $\tilde{H}$ [Eq. (3)] is equal to this probability times the number of observations over which the number of close returns hits is counted, $n$:

$$\tilde{H} = np(\text{black}). \quad (3)$$

The sample $\chi^2_c$ is expressed as Eq. (4)

$$\chi^2_c = \frac{\sum (H_i - \tilde{H})^2}{np(1-p)}. \quad (4)$$

The calculated value, $\chi^2_c$, is then compared to the test $\chi^2_i$. If $\chi^2_c > \chi^2_i$, the null hypothesis that the data are iid is rejected. For the Henon series the calculated $\chi^2$ is 900.45, while the $\chi^2$ for $300 - 1$ degrees of freedom and a 5% significance level is 340.33. Therefore, the null hypothesis that the data are iid is rejected. The pseudorandom series has a $\chi^2_c$ of 267.22; therefore, the null of iid cannot be rejected. The reliability of this test was established using simulation studies involving iid, chaotic, and autoregressive series (Gilmore, 1996).

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2 This formulation is obtained from Bury (1975), example 6.7, p. 199.
3.2. Data

The data consist of daily exchange rate series for the following currencies: British pound, German mark, and Japanese yen, covering January 7, 1976–December 1, 1995, for the pound and the mark, and January 7, 1976–June 17, 1994 for the yen. The rate of change is calculated as \(100 \times \ln(S_t/S_{t-1})\), giving 5197 observations for the pound and the mark and 4814 observations for the yen.

3.3. Results

The qualitative close returns test was applied to each series to look for evidence of chaotic structure. For each currency a series of close returns plots was created, covering the entire range of the data. Fig. 5 presents a close returns plot of the first 800 observations for each series, covering the period January 6, 1976–January 26, 1979. As can be seen from these plots, there is no evidence to indicate the presence of chaos; that is, there is no pattern of horizontal line segments revealing a recursive behavior in the series. Plots for all the subsequent time periods, as well as day-of-the-week plots also failed to indicate possible chaotic behavior. These results do not, therefore, support the tentative evidence of chaos in the British pound (for the period 1973–1981 and 1973–1990) and for the Japanese yen (1973–1990) found by De Grauwe et al. (1993), using the correlation dimension and the BDS tests. It is also useful to note that the methods used by De Grauwe et al. (1993) produced aggregated results which in a few cases appeared to indicate that chaotic behavior could have occurred somewhere within the time periods cited. The close returns plots, however, do not aggregate the information in the data but instead would identify the location of chaotic “episodes” within a time series, if present.

What these plots do reveal, however, is evidence of some type of structure in the data, as well as changes in the behavior of each series over time. For the interval shown in Fig. 5 the plots show a strong pattern of changes in volatility, with periods of relatively low volatility (black areas) broken by periods of higher volatility (white areas). The later plots, not shown, still indicate some structure, but it is less pronounced.\(^3\)

Results of the quantitative form of the close returns test, which are shown in Table 1, reinforce the evidence from the plots. For ease of exposition the results are shown in ratio form, \(\chi^2 / T\), so a ratio greater than 1.0 indicates rejection of the null hypothesis. In each case the \(\chi^2\) ratio for the entire series exceeds 1.0, leading to rejection of the iid null and indicating the presence of some type of dependence. Because the plots indicate changes in the behavior of each series over time, \(\chi^2\) ratios were also calculated on each half of the data sets. For each, the \(\chi^2\) continued to reject the iid null for the first half of the data but did not for the second half.

\(^3\) The results, not shown here, are available on request from the author.
Fig. 5. Close returns plots of exchange rate series. This figure presents close returns plots of the first 800 observations for the British pound, the German mark, and the Japanese yen.

The results of the $\chi^2$ test for the later time period will be analyzed further. It will be recalled that the close returns plot is specifically designed to search for chaos by
identifying the recurrence property. The projection in the horizontal direction to construct the histogram for the quantitative test emphasizes the regularly spaced horizontal line segments which result from that property. If the data have structure which is not chaotic, this one-dimensional projection can “wash out” other types of signal. The structure that is not chaotic may be more easily detected by applying the \( \chi^2 \) test directly to the two-dimensional plot itself to detect departures from iid.

To make the computations simpler, we can construct the two-dimensional plot as a square graph with \( x_i \) versus \( x_j \); the information is identical. The plot is then subdivided into boxes of \( n \) observations on a side. The number of close returns is counted up for each box, the probability of a black point and the expected count are computed as previously, and the \( \chi^2 \) ratio is then calculated. Using this approach, the \( \chi^2 \) ratios for the later time period for each exchange rate series were recalculated, using a range of box sizes from \( n = 20 \) to \( n = 50 \). These results are shown in Table 2. All of the \( \chi^2 \) ratios now exceed 1.0, leading to rejection of the iid null. Some type of dependence therefore appears to be present throughout each exchange rate series. We cannot conclude at this point that it is nonlinear dependence, since the \( \chi^2 \) test is sensitive to both linear and nonlinear departures from iid.

Table 2
Box-plot \( \chi^2 \) ratios for later time period, exchange rate series

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<tr>
<td>Box size</td>
<td>20</td>
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<td>30</td>
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<tr>
<td></td>
<td>2.93</td>
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<td>4.79</td>
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<tr>
<td></td>
<td>9.95</td>
<td>9.95</td>
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Table 3
Box-plot $\chi^2$ on residuals of yen linear model

<table>
<thead>
<tr>
<th>Box size</th>
<th>$\chi^2$ ratio</th>
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</thead>
<tbody>
<tr>
<td>20</td>
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<tr>
<td>25</td>
<td>3.16</td>
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<td>4.38</td>
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<tr>
<td>45</td>
<td>7.44</td>
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<tr>
<td>50</td>
<td>4.74</td>
</tr>
</tbody>
</table>

Focusing on the second time period, we filtered the data by the following autoregression to account for possible linear dependence [Eq. (5)]:

$$x_t = b_0 + b_1 D_{M,t} + b_2 D_{T,t} + b_3 D_{W,t} + b_4 D_{Th,t} + b_5 D_{h,t} + \sum b_i x_{t-i} + \epsilon_t$$  \hspace{1cm} (5)

where $D_{M,t}$, etc., are dummy variables for the days of the week and holidays (excluding weekends). The $\chi^2$ ratio applied to the model standardized residuals exceeded 1.0 for the pound and the mark, 1.211 and 1.013, respectively, but not for the yen (0.965). However, when the box plot $\chi^2$ ratios were calculated on the yen residuals, the ratios also exceeded 1.0 (Table 3). Therefore, there are indications of nonlinear dependence for all three series.

To model the nonlinear dependence each exchange rate series was then fit with a generalized autoregressive conditional heteroscedasticity (GARCH) model, specified as follows:

$$x_t = b_0 + b_1 D_{M,t} + b_2 D_{T,t} + b_3 D_{W,t} + b_4 D_{Th,t} + b_5 D_{h,t} + \sum b_i x_{t-i} + \epsilon_t$$

where $\epsilon_t$ is normally distributed, with zero mean and variance $h_t$, such that [Eq. (6)]

$$h_t = \gamma_0 + \gamma_1 h_{t-1} + \phi \epsilon_{t-1}^2.$$  \hspace{1cm} (6)

Applying the Ljung-Box $Q_5(k)$ and $Q_{xx}(k)$ tests to the standardized residuals, we find that the GARCH models fit these data reasonably well. However, a $\chi^2$ of 1.038 for the British pound indicates remaining nonlinear structure. The $\chi^2$ for the mark and the yen were 0.980 and 0.899, respectively, leading to failure to reject the iid null for the standardized residuals. But the box-plot form of the $\chi^2$ test detects evidence of nonlinearity in each series (Table 4), which is not accounted for by the GARCH models. Each data set was also fitted to an exponential GARCH (EGARCH) model, producing comparable results, with the box plots continuing to reject the null hypothesis of iid. These findings are consistent with Hsieh (1989a), who found evidence of nonlinearity in the standardized residuals of GARCH.

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4 The lag length for the AR($m$) terms was chosen so that the $Q_5(50)$ test is not significant at the 5% level ($m = 6$, 0, 6, for the pound, mark, and yen, respectively).

5 The results are available on request from the author.
models, using the BDS test. They also agree with the Fernandes (1998) finding that GARCH(1,1) and EGARCH(1,1) models do not adequately capture the nonlinearity in the British pound.

4. Conclusions and recommendations for future work

The close returns test is a powerful tool to detect evidence of chaotic behavior or other structure in time series data. The qualitative form of the tests uses a fundamental characteristic of chaos, the recurrence property, to identify chaotic behavior. Quantifying this test involves the tradeoff of using the iid null hypothesis, with the alternative including both linear and nonlinear structure.

Application of these tests to daily exchange-rate series supports earlier research finding nonlinear dependence. The results of the qualitative close returns test do not support the findings of De Gruwe et al. (1993) of possible chaos in the British pound or the Japanese yen, instead agreeing with the conclusion of Brock et al. (1991) and Hsieh (1989b) that there is no evidence of chaotic behavior. Application of the quantitative $\chi^2$ tests to the standardized residuals of GARCH and EGARCH models was largely consistent with Brock et al., Fernandes (1998), and Hsieh. The box-plot $\chi^2$ ratios for the GARCH and EGARCH models, compared to those for the unfiltered and linearly filtered series, indicate that the models have captured some, although not all, of the nonlinearity dependence, at least for the mark and the yen; GARCH and EGARCH were less successful for the pound, again consistent with other studies.

Further research needs to proceed in several directions. First, while GARCH(1,1) and EGARCH(1,1) models are commonly applied to financial time series, such as exchange rates, the present results point to the need to develop more complex models to capture the remaining nonlinearity present in these series. Work is proceeding in this area. Second, all the methods which have been developed to date to test for chaos can at best provide indications of low-dimensional chaos, that is, behavior resulting from a fairly simple model. Yet, there is no reason to assume that financial behavior is the result of such simple models. Consequently, further theoretical advances are required to establish the basis for tests to detect more complex forms of chaotic behavior.
References


