Emergence of order from chaos: A phenomenological model of coupled oscillators

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This paper aims to study the transition to order from chaos using a mathematical model of coupled non-identical oscillators. In the course of our analysis, we adopt a different control parameter other than the conventional parameters: coupling strength and frequency-mismatch, generally used in literature to obtain the aforementioned transition. Both the interacting oscillators become periodic, and they lead to the synchronized state as the control parameter changes monotonically. Initially, the participating oscillators, with low amplitude of oscillations, are in the chaotic desynchronized state. Besides, during the periodic oscillations, the oscillators exhibit high amplitude of oscillations. We illustrate the corresponding results using two examples of coupled oscillators. Such emergence of order from chaos with an accompanying increase in amplitude is ascertained in various numerical simulations and experimental observations.

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1. Introduction

The period-doubling route to chaos is a well-documented topic in the literature of dynamical systems and a part of the elementary books on nonlinear dynamics [1–6]. The opposite scenario, i.e., the emergence of order in dynamical systems from chaos, is also an important phenomenon occurring in nature and is observed in various physical systems [7]. Similar qualitative change in the dynamics of any dynamical system with the variation of its parameter is termed as bifurcation [6]. Bifurcation is detected in various theoretical studies and experimental observations of the dynamical systems [8]. Besides, there exists a different kind of bifurcation called blowout bifurcation, detected in coupled lasers, implies the loss of synchronization [9].

In the case of unidirectionally coupled chaotic oscillators arranged in a ring-like structure, the transition to order from chaos is detected by varying the frequency mismatch as the required control parameter [10]. For coupled identical Lorenz oscillators, by controlling the coupling strength parameter, periodic orbits are obtained from the chaotic dynamics [11]. Similarly, the transition between order and chaos are ascertained in interacting van der Pol oscillators after varying the coupling strength parameter [12].

The transition to synchronization from desynchronization in coupled van der Pol oscillators has been investigated, and the coupling strength parameter and the frequency mismatch are used as the required control parameters for bifurcation [13]. In a system of periodically forced coupled non-identical Duffing oscillators, order and chaos have been discussed using the amplitudes and the frequencies of the periodic terms as control parameters [14]. Here, we study the emergence of order from chaos in a system of coupled non-identical oscillators; in order to scrutinize the mentioned transition, we use one of the most important phenomena of nonlinear dynamics: synchronization.

Synchronization, the coordinated motion of the interacting dynamical systems, is a universal phenomenon [15]. Synchronization is studied mainly in the presence of weak coupling among the interacting systems such that the dynamics of the participating systems do not vary significantly after introducing the coupling term [15]. Too strong coupling may make the coupled systems unified [15]. In interacting dynamical systems, we can observe different kinds of synchronizations such as phase synchronization, generalized synchronization, lag synchronization, complete synchronization, measure synchronization [15–19]. For the purpose of this paper, we briefly discuss phase synchronization and generalized synchronization in the context of two coupled oscillators. In the case of phase synchronization, phase locking is ascertained, i.e., the phase difference of the interacting oscillators becomes con-
stant [20]. Besides, for generalized synchronization, the phase variables of one oscillator become a function of the phase variables of the second oscillator [21].

To summarize, in many cases, the discussed transition to order from chaos in coupled oscillators is scrutinized by varying either the coupling strength parameter or the frequency mismatch parameter. In some cases, noise induces the transition to order from chaos in coupled systems [22–25]. However, in experiments, sometimes there are more constraints on varying the coupling strength; instead, it may be natural to change another system parameters [26–29]. Therefore, motivating from the experimental observations, this paper aims to study the aforesaid transition from a different perspective: by changing one of the system parameters other than the coupling strength parameter or the frequency mismatch parameter. We use a simple phenomenological model of mutually coupled non-identical oscillators and scrutinize the dynamics of interacting oscillators after varying one of the system parameters monotonically. Also, the diffusive coupling scheme is adopted to couple the oscillators since many of the experimental observations are modelled using interacting non-identical oscillators coupled diffusively [15,30]. In the course of our investigation, we choose two examples of coupled non-identical oscillators to illustrate the results.

For the first example, generalized synchronization between the interacting oscillators is achieved from the desynchronized state as the bifurcation parameter increases monotonically. At intermediate values of the bifurcation parameter, we detect intermittent phase synchronization and phase synchronization. Along similar lines, in the case of second example, as we increase the bifurcation parameter monotonically, the transition from a desynchronized state to a state of generalized synchronization is observed through the route of phase synchronization between the interacting oscillators. We emphasize that in both examples, the transitions are associated with low amplitude chaotic dynamics of the participating oscillators at the initial stage and high amplitude periodic oscillations at the final stage. In the next section, we illustrate the corresponding results of this paper in detail.

2. Results

We choose two examples of interacting autonomous three dimensional oscillators: coupled non-identical Lorenz oscillators [31] and interacting non-identical dynamo oscillators [32]. In the first half of this section, we discuss the results corresponding to coupled Lorenz oscillators, and the second half contains the results of coupled dynamo oscillators.

2.1. First example: coupled non-identical Lorenz oscillators

For coupled non-identical Lorenz oscillators, the explicit form of equations of motion using z-coupling are as follow:

\[
\begin{align*}
\frac{dx_1}{dt} &= 10(y_1 - x_1), \\
\frac{dy_1}{dt} &= x_1(r - z_1) - y_1, \\
\frac{dz_1}{dt} &= x_1y_1 - \frac{8}{3}z_1 + \alpha(z_2 - z_1), \\
\frac{dx_2}{dt} &= 9.9(y_2 - x_2), \\
\frac{dy_2}{dt} &= x_2(r - z_2) - y_2.
\end{align*}
\]  

where \((x_i, y_i, z_i)\) are the phase space variables of the \(i\)th oscillator with \(i = 1, 2\). The scalar \(\alpha\) measures the coupling strength between the two oscillators, and \(r\) is one of the system parameters. One can clearly identify the parameter mismatch in Eqs. (1a) and (1d). The coupling strength parameter \(\alpha\) is chosen as 0.65. Now we change the parameter \(r\) monotonically within the range [28, 164], and study the corresponding change in the dynamics of the coupled system.

In Fig. 1a, at \(r = 28\), the time series \(z_1(t)\) is aperiodic, and the amplitude of oscillations varies within the approximate range [3.96, 45.04]. Also, both the time series lead to the desynchronized state at \(r = 28\). We discuss this desynchronization in detail at the later part of this section (Fig. 3a). At an intermediate value, \(r = 142.9\), phase synchronization is ascertained (Fig. 2). In order to detect the aforesaid phase synchronization, phases \(\phi_i\) of the individual systems of Eq. (1) are calculated using the following procedure [15,20]:

\[
z_i' := \sqrt{(x_i^2 + y_i^2)} \quad \text{and} \quad \phi_i := \tan^{-1} \left( \frac{z_i}{x_i} \right).
\]

The corresponding phase difference \(\phi_1 - \phi_2\) (in degree) is plotted with \(r\) for two different values of \(r\) in Fig. 2. In Fig. 2a, the phase difference \(\phi_1 - \phi_2\) occasionally becomes zero at \(r = 141\); hence we call this state as the intermittent phase synchronized [26,33] state between the interacting oscillators. Further increase in \(r\), at \(r = 142.9\) (Fig. 2b), leads to the phase locking of the
oscillators. Hence, both the oscillators exhibit the phase synchronized state at $r = 142.9$. This phase synchronization remains up to $r = 164$.

Finally, at $r = 164$, as shown in Fig. 1b, periodic dynamics are detected and the amplitudes of $z_1(t)$ and $z_2(t)$ vary within the approximate ranges $[106.76, 221.29]$. The time series $z_1(t)$ and $z_2(t)$ are separated by a constant time lag with a slight mismatch in the magnitudes at peaks. Thus, one can write $z_1(t)$ as a function of $z_2(t)$. Consequently, interacting non-identical Lorenz oscillators attain generalized synchronization state at $r = 164$. Note that in Section 2.1.1, using the recurrence theory [26,34], we confirm the existence of aforementioned generalized synchronization between the oscillators. Also, using the recurrence theory, desynchronized state and phase synchronized state can be confirmed between two bidirectionally coupled non-identical oscillators [26,34]. We remark that instead of using Eq. (2), calculation of phases of the coupled oscillators (Eq. (1)) using the standard Hilbert transformation [15,20] technique eventually boils down to the unchanged results as depicted in Fig. 2.

2.1.1. Tool: comparison of probability of recurrences

In this section, we discuss the framework of recurrence plots in details [34]. Let $[u_i]_{i=1}^{N}$ be the given time series, then the corresponding reconstructed vectors are:

$$v_k = (u_k, u_{k+d}, ..., u_{k+(d-1)r_0}),$$

where $k = 1, 2, ..., N_1$, and $N_1 := N - (d - 1)r_0$. Here, $r_0$ and $d$ are the required time delay and minimum embedding dimension respectively, chosen appropriately (See Appendices A and B for detail description). Furthermore, the recurrence plots are based on the matrix defined as follows [34]:

$$R_{k,l} = \Theta(\varepsilon - ||v_k - v_l||),$$

where $k, l = 1, 2, ..., N_1$, $\tau$ is the time delay, $||...||$ represents the standard Euclidean norm, and $\varepsilon$ is a scalar. $\Theta(x)$, the Heaviside function, is 1 if $x \geq 0$; 0 for $x < 0$. Therefore, we obtain a matrix of dimension $N_1 \times N_1$ with elements 0 and 1. Finally, we define the probability of recurrence $P(\tau)$ as follows:

$$P(\tau) = \frac{\sum_{m=1}^{N_1-r} R_{m,m+\tau}}{N_1 - \tau} = \frac{\sum_{m=1}^{N_1-r} \Theta(\varepsilon - ||v_m - v_{m+\tau}||)}{N_1 - \tau}.$$  

$P(\tau)$ measures the probability that the phase space trajectory arrives to the $\varepsilon$-neighbourhood of the point $v_k$ after time interval $\tau$.

Phase synchronization and generalized synchronization can be ascertained after comparing the aforesaid probabilities for both the oscillators. If the probabilities overlap and the peaks in $P(\tau)$ are observed at the same values of $\tau$ imply the phase locking of the interacting oscillators. Along with this phase locking, if the peaks of $P(\tau)$ corresponding to both the oscillators have identical heights, then the two oscillators are said to be in the generalized synchronization state. The heights of the peaks of $P(\tau)$ are related to the amplitude of the oscillator. Lastly, if the corresponding probabilities do not overlap, both the oscillators lead to the desynchronized state.

In Fig. 3, we have plotted the probabilities $P(\tau)$ with different values of $\varepsilon$ at three distinct values of $r$: 28 (Fig. 3a), 142.9 (Fig. 3b), and 164 (Fig. 3c) for the interacting Lorenz oscillators (Eq. (1)). The corresponding probabilities are non-overlapping at $r = 28$. Therefore, Fig. 3a supports the existence of desynchronized state between the interacting Lorenz oscillators at $r = 28$. By contrast, Fig. 3b shows the phase synchronization while Fig. 3c depicts the generalized synchronization between the interacting oscillators.

2.1.2. Bifurcation diagrams

To understand the dynamics of the coupled oscillators (Eq. (1)), we have plotted the bifurcation diagrams [6] of the individual Lorenz oscillator in Fig. 4. The bifurcation diagrams are obtained by plotting the local maxima of the time series after removing the transient (initial 10% data points) corresponding to a particular value of the parameter $r$.

In both the bifurcation diagrams of Fig. 4, both the oscillators of Eq. (1) change their qualitative dynamics with increase in $r$. The initial chaotic dynamics becomes periodic at higher $r$, i.e., bifurcation occurs, and $r$ is the required bifurcation parameter. The amplitudes of oscillations of both the oscillators are smaller during the chaotic dynamics than that of during periodic motion.

2.2. Second example: coupled non-identical dynamo oscillators

Now we extend our study to the second example: coupled non-identical dynamo oscillators. The equations of motion using $x$-coupling are given by:

$$\frac{dx_1}{dt} = y_1z_1 - \rho x_1 + \alpha(x_2 - x_1),$$

$$\frac{dy_1}{dt} = (z_1 - 0.5)x_1 - \rho y_1.$$
identical oscillators and use bifurcation diagrams to understand the dynamics of the individual oscillator. We have studied this transition from a different point of view, keeping the conventional control parameters such as coupling strength parameter and frequency mismatch unaltered, instead varying a different systems parameter. Consequently, we have obtained generalized synchronization through the route of intermittent phase synchronization and phase synchronization for bidirectionally coupled non-identical Lorenz oscillators. In the course of our analysis, we have increased one system parameter monotonically, keeping all other parameters unaltered. During the desynchronized state, the amplitudes of oscillations are smaller compared to that in case of the synchronized state. Besides, for two mutually coupled non-identical dynamo oscillators, generalized synchronization is reached from the desynchronized state through the route of phase synchronization. Similar to the example of coupled non-identical Lorenz oscillators, one may draw the bifurcation diagrams of interacting dynamo oscillators to detect a similar transition from chaotic to periodic dynamics. It is worthy mentioning that although in this paper all the conclusions are drawn using the z-coordinates of Eq. (1) (or Eq. (6)), the consequences remain unchanged if one deals with either x or y-coordinates of Eq. (1) (or Eq. (6)).

In experiments, researchers have found similar transitions in thermoacoustic [26], aeroacoustic [35], and aeroelastic systems [28]. Here, we discuss one experimental observation briefly. In thermoacoustic systems, one of the simplest phenomenological models that describe the thermoacoustic instability consists of two mutually coupled non-identical oscillators [26]. Two interacting oscillators physically represent acoustic pressure and heat release rate oscillations of the turbulent combustor. Using synchronization theory, the thermoacoustic instability from the combustion noise [26]. The state of combustion noise manifests as small-amplitude aperiodic oscillations of the outputs: acoustic pressure and heat release rate. Also, both the oscillators lead to the desynchronized state. On the contrary, during the occurrence of thermoacoustic instability, the coupled oscillators exhibit large-amplitude periodic oscillations, and both of them have coordinated motions, which indicates the existence of synchronization between

\[
\frac{dz_1}{dt} = 1 - x_1y_1, \quad (6c)
\]

\[
\frac{dx_2}{dt} = y_2z_2 - (\rho + \Delta \rho)x_2 + \alpha(x_1 - x_2), \quad (6d)
\]

\[
\frac{dy_2}{dt} = (z_2 - 0.5)x_2 - (\rho + \Delta \rho)y_2, \quad (6e)
\]

\[
\frac{dz_2}{dt} = 1 - x_2y_2. \quad (6f)
\]

We choose the coupling strength as \(\alpha = 0.65\) and \(\Delta \rho = 0.1\), and vary the system parameter \(\rho\) monotonically from 0.7 to 5.7 to investigate the corresponding change in the dynamics. Similar to the previous example of coupled Lorenz oscillators, as the bifurcation parameter \(\rho\) increases monotonically, the ranges of oscillations of the z-coordinates of the interacting oscillators enhance (Fig. 5).

Additionally, the initial aperiodic behavior of \(z_i(t)\) becomes periodic at higher \(\rho\). Both the oscillators lead to the desynchronized state at \(\rho = 0.7\) (Fig. 6a). As the bifurcation parameter \(\rho\) is further increased, \(\rho = 3.7\), the participating oscillators are in phase synchronized state, as shown in Fig. 6b. Finally, at \(\rho = 5.7\), both the oscillators exhibit generalized synchronized state (Fig. 6c). Therefore, similar to the previous example, we obtain generalized synchrony between the coupled oscillators from a desynchronized state through the route of phase synchrony.

Note that throughout this paper, all the results are illustrated with \(\alpha = 0.65\). Interestingly, one may work with other values of \(\alpha\). For example, we have plotted the bifurcation diagrams at \(\alpha = 0.7\) (refer to Appendix C). All the conclusions drawn from Figs. 2, 3, 5, and 6 remain unaltered at \(\alpha = 0.7\). Furthermore, we got invariant conclusions at \(\alpha = 0.3, 0.4, 0.55, \) and 0.6.

3. Conclusions and discussions

In conclusion, we have studied the transition to order from chaos using a simple phenomenological model of coupled non-
the oscillators. In order to get the transition, the authors have used the mean velocity of the flow as the required control parameter and varied it monotonically. In the course of the transition to thermoacoustic instability, at intermediate values of the mean velocity of the flow, intermittent phase synchronization and phase synchronization have been ascertained between acoustic pressure and heat release rate.

Besides, there exists the occasional coupling induced synchronization [36]—where the systems interact occasionally instead of continuously and synchronization is observed. Generally, using the occasional coupling synchronization is detected at higher coupling strength parameter values where the continuous coupling fails to lead synchrony [37–42]. Scrutinizing the discussed transition to order from chaos using the occasional coupling in coupled oscillators model may be an interesting future direction to pursue.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Choosing of the time delay

First we have to focus on choosing the appropriate \( \tau_0 \) to construct the reconstructed vectors \( \{u_j\}_{j=1}^N \) from the given time series \( \{u_j\}_{j=1}^N \), then we shift our interest to calculate the minimum embedding dimension. To find \( \tau_0 \), we calculate the autocorrelation function defined as follows:

\[
A(\tau_0) = \frac{\sum_{j=1}^{N-N_0}(u_j - \mu)(u_{j-N_0} - \mu)}{\sum_{j=1}^{N}(u_j - \mu)^2},
\]

where \( \mu \) is the arithmetic mean of \( \{u_j\}_{j=1}^N \), and \( N_0 \) is that particular element in \( \{u_j\}_{j=1}^N \) separated by time gap \( \tau_0 \). For coupled non-identical Lorenz oscillators, we have calculated \( A(\tau_0) \) for different values of \( \tau_0 \), and choose that particular \( \tau_0 \) at which \( A(\tau_0) \) changes its sign from positive to negative [43]. From Fig. A.7a and b, it is observed that for both the interacting Lorenz oscillators the required \( \tau_0 \) are approximately 0.08.

There is another well-known method to determine the time delay [44]. In this method, one may calculate the mutual information (\( I \)) between two temporally separated data points \( u_i \) and \( u_{i+\tau_0} \) of the given time series \( \{u_j\}_{j=1}^N \). Finally, choose the corresponding time delay \( \tau_0 \) at which the mutual information function has its first minimum [44]. In order to implement this method, we calculate mutual information using the technique [45] elaborately discussed in next paragraph.

At the beginning, we define two variables \( s \) and \( q \) such that

\[ [s_j, q_j] := [u_j, u_{j+\tau_0}]. \]

Furthermore, we define the entropy as follows:

\[
H(s) = -\sum_j p(s_j) \log_2 (p(s_j)),
\]

where \( \{p(s_j)\}_{j=1}^N \) are the probabilities associated with the variables \( \{s_j\}_{j=1}^N \). In other words, \( p(s) \) is the probability mass function for a discrete distribution, and represents the probability density function for a continuous distribution. In Eq. (A.2), we choose the logarithmic function to base 2 so that \( H(s) \) has unit in bit. Similar to Eq. (A.2), we can define entropy \( H(q) \) corresponding to \( q \). Finally, we define the mutual information between \( s \) and \( q \) as follows:

\[
I(s, q) = H(s) - H(s|q),
\]

where \( H(s|q) \) is the conditional entropy of \( s \) on the knowledge of \( q \). \( H(s|q) \) measures the entropy of \( s \) conditional on the known values of \( q \). Furthermore, mutual information \( I(s, q) \) is symmetric in \( s \) and \( q \), i.e., \( I(s, q) = I(q, s) \) [45].

Following this method, the calculated mutual information functions for the \( z \)-coordinates of the interacting Lorenz oscillators are plotted with different time delays in Fig. A.7c and d. The first minima of \( I \), in both the cases, are at \( \tau_0 = 0.08 \). Therefore, in conclusion, using both the methods we obtain the same time delay for the reconstructed vectors.

Appendix B. Calculation of the minimum embedding dimension

Now, we find the required minimum embedding dimension using the method given by Cao [46]. Following the aforesaid method, we define two measures \( E_1(d) \) and \( E_2(d) \)—which incorporate the intuition of false nearest neighbour [47]—as explicit functions of embedding dimension \( d \). The first measure \( E_1(d) \) by definition, becomes constant after a particular \( d \) if the given data set is coming from an attractor. Besides, \( E_1(d) \) keeps increasing with increase in \( d \) if the given data set consists of random numbers. However, sometimes, because of the smaller size of the given data set, \( E_1(d) \) keeps on increasing with increase in \( d \) although the time series is coming from an attractor. To overcome this problem, the second measure \( E_2(d) \) is introduced. By definition of \( E_2(d) \), it is always unity for random data and is independent of \( d \). However, \( E_2(d) \) is increasing from lower to higher values as \( d \) is increasing for any deterministic data.

In Fig. B.8, \( E_1(d) \) are plotted with \( \tau_0 = 0.08 \) for both the oscillators. In both the cases, \( E_1(d) \) stop changing after \( d = 3 \), indicating the minimum embedding dimensions are 3. Since \( E_2(d) \) varies from a smaller to larger values as \( d \) increases monotonically, these variations support that the time series are generated from deterministic systems.

For the second example: coupled non-identical dynamo oscillators, following the discussed methods, we obtain the required time
delay and minimum embedding dimension as 0.08 and 4 respectively. Although we have not given the plots to calculate the time delay and the minimum embedding dimension, one may always get plots similar to Figs. A.7 and B.8 for the second example.

Appendix C. Bifurcation diagrams at different coupling strength parameter

Bifurcation diagrams are plotted for z-coupled non-identical Lorenz oscillators at coupling strength parameter $\alpha = 0.7$ (Fig. C.9). It is clear that chaotic (with smaller amplitudes of oscillations) and periodic (with larger amplitudes of oscillations) dynamics are detected respectively for lower and higher values of the bifurcation parameter $r$.

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