Detecting phase synchronization between coupled non-phase-coherent oscillators

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Abstract

We compare two methods for detecting phase synchronization in coupled non-phase-coherent oscillators. One method is based on the locking of self-sustained oscillators with an irregular signal. The other uses trajectory recurrences in phase space. We identify the pros and cons of both methods and propose guidelines to detect phase synchronization in data series.

1. Introduction

Since the first studies on synchronization of chaotic systems by Fujisaka and Yamada [1], and Pecora and Carroll [2], chaos synchronization has become one of the most active research fields in nonlinear dynamics. Initially the attention was centered on what is known today as complete synchronization [3]. The trajectories of two properly coupled identical chaotic systems will converge to a common path, and will remain together, regardless of differences in their initial conditions. A more particular type of synchronization is of interest here. We focus on phase synchronization (PS), where two coupled chaotic processes develop synchronism of their phases while their amplitudes continue to evolve independently [4,5]. PS appears quite often in nature, especially because it is very difficult or even impossible to find two identical natural systems and also because it is related to weak interactions among systems. This has been observed in various natural systems and laboratory experiments, in neuronal systems, ecological systems, laser array and electromechanical systems, amongst others [6–14]. Despite being a widely spread phenomenon, PS, particularly in data series may not be easily detected due to noise, nonstationarity and limited duration. This is exactly what we are interested in here, the problem of detecting PS in data series.

Phase \( \phi(t) \) is a concept originally introduced for periodic oscillators with dynamics represented in phase space by limit cycles. It is a monotonously increasing function of time. The extension of this concept to chaotic oscillators is not necessarily straightforward and a number of approaches have been proposed. In some cases it is possible to find a suitable projection of the attractor so that the trajectory circles around some point taken as the center of rotation. This is the case depicted in Fig. 1(a). From this projection, the phase can be identified with the angle of rotation with respect to a fixed axis of reference. The fluctuations of the phase around the average can be characterized by a diffusion coefficient \( D_\phi \). This coefficient measures the degree of phase coherence of the chaotic signal. A very small \( D_\phi \) indicates that the phase dynamics is relatively regular and the power spectrum of the signal shows a sharp
peak on top of a broadband component. This situation is known as the phase-coherent regime. For this regime, the approaches introduced to measure phase for a chaotic system work reasonably well, despite the presence of some sporadic regions in which the phase does not increase uniformly or even decreases. However, there are other situations for which it is not possible to define a projection of the phase space so that the whole chaotic trajectory on this projection rotates around the same point. This situation is represented in Fig. 1(b). If a phase is somehow introduced, in general it results in a large value of $D_0$, which indicates that the phase is not coherent. The power spectrum of the signal presents just a broadband component. This dynamic situation is known as non-phase-coherent regime. For this regime, the majority of the proposed methods for measuring phase do not work. The "phase" that they generate is not a monotonously increasing function of time [15] or presents ambiguity [16].

Two methods have recently been proposed for detecting and quantifying PS in coupled chaotic systems. Both methods are applicable to phase coherent and also to rather complex, non-phase-coherent regimes and experimental data. They do not require the determination of a phase associated with trajectories. The first method was introduced by Rosenblum and co-workers [17] and is based on the locking of an ensemble of standard periodic self-sustained oscillators by an irregular signal. This method is known as Locking-Based Frequency Method (LBFM). The other method, proposed by Romano and coworkers [18], is based on the recurrence of the system's trajectory into the neighborhood of a former state. This method is named Phase Synchronization by Means of Recurrence (PSMR), and can be considered as an extension of the method so-called "r-recurrence rate", that was conceived to detect complete synchronization [19]. Here we compare these two methods by identifying conditions for optimum performance of each one in detecting PS.

The next two sections describe and test the LBFM and the PSMR methods sequentially. The fourth section presents our test results. Next, we compare the outcome of both methods when applied to an experimental data series. The last section concludes and provides guidelines related to the use of both methods.

2. Locking-Based Frequency Method (LBFM)

This method is based on the use of an ensemble of auxiliary limit cycle oscillators. It considers an ensemble of uncoupled limit cycle oscillators with natural frequencies $\omega_k$ distributed in an interval $[\omega_{\text{min}}, \omega_{\text{max}}]$. Each oscillator is driven by a common periodic frequency with frequency $\epsilon \in [\omega_{\text{min}}, \omega_{\text{max}}]$. This force synchronizes those elements of the ensemble which have frequencies close to $\epsilon$. When the frequencies of the driven limit cycle oscillators $\Omega_k$ are plotted vs. the natural frequencies $\omega_k$, the synchronization manifests itself in the appearance of a horizontal plateau, where the frequency of entrained elements is equal to $\epsilon$. An unknown frequency of the drive can be revealed by the analysis of the $\Omega_k$ vs. $\omega_k$ plot. Following [17], we can determine the unknown frequency from this plateau by finding $\hat{\omega}_k^2$, which is the minimum of the running variance $\sum_{j=k-L}^{k+L} \Omega_j - \hat{\omega}_k^2)^2$, where $\hat{\omega}_k = (2L + 1)^{-1} \sum_{j=k-L}^{k+L} \Omega_j$ and $L = 10$. This quantity can be identified as the middle point of the plateau.

Based on this idea, the authors conceive their method: Consider an ensemble of Poincaré oscillators driven by a signal $X(t)$ that one wants to analyze.

$$\hat{\lambda}_k = (1 + i\omega_k)\hat{A}_k - |A_k|^2 A_k + \epsilon X(t),$$

where $A = Re^{i\phi}$ is the complex amplitude. Replacing $\hat{A} = \hat{R}e^{i\phi} + \hat{R}\hat{\phi}e^{i\phi}$ in Eq. (1) we have

$$\hat{R} + \hat{R}\phi = (\omega_k R - \epsilon X(t) \sin \phi)\hat{i} + R + R^3 + \epsilon X(t) \cos \phi.$$  

Separating the real amplitude $R$ and the phase $\phi$ from the complex amplitude we obtain

$$\dot{R}_k = R_k - R_k^2 + \epsilon X(t) \cos \phi_k,$$

$$\dot{\phi}_k = \omega_k - R_k^{-1} \epsilon X(t) \sin \phi_k.$$  

If, for small values of $\epsilon$ the amplitude $R$ is close to unity and omitting its fluctuations, we have

$$\dot{\phi}_k = \omega_k - \epsilon X(t) \sin \phi_k,$$

as phase equations for the forced oscillators. Thus, the observed frequencies are given by

$$\Omega_k = \lim_{t \to \infty} \left[ \phi_k(t) - \phi_k(0) \right]/t,$$

where $k$ indicates the order of the oscillator in the ensemble.

According to Ref. [17], this approach can be used to determine if two chaotic oscillators are phase synchronized, even if they are in non-phase-coherent regime. To accomplish this, the scalar signals $x_{1,2}$ from these systems are used as input for two identical ensembles of limit cycle oscillators. For each one, we obtain the plots $\Omega_{k,1,2}$ vs. $\omega_{k,1,2}$. If both plots present plateaus over a common range of $\omega_{k,1,2}$, it can be claimed that there is PS between the oscillators within that common range [17]. In our computations the signal $X(t)$ is normalized to have zero mean and unit variance so $\epsilon$ is the only parameter of the method. For quantitative evaluation purposes, gauge criterion can be defined to evaluate and compare the plateaus.

3. Phase Synchronization by Means of Recurrence (PSMR)

The PSMR method exploits Poincaré’s [20] concept of recurrence: A dynamic system will return arbitrarily close to each point of its past trajectory after a sufficiently long time interval.

Given a dynamic system’s trajectory $[x_i]_{i=1}^{N}$, it can be said that at time $t = 0$ the trajectory has returned to the close neighborhood of a former state at time $t = 0$. If

$$R_{i,j}^{(e)} = \Theta(\epsilon - ||x_i - x_j||) = 1,$$

where $\epsilon$ is a pre-defined threshold, $\Theta(\cdot)$ is the Heaviside function and $\delta t$ is the time interval between consecutive samplings. Based on this definition, it is possible to estimate the probability $P^{(e)}(\tau)$ that the system will return to the neighborhood of a former point $x_i$ of the trajectory after $\tau$ time steps, where the neighborhood is defined as a box of size $\epsilon$ around the state $x_i$. That is

$$P^{(e)}(\tau) = \frac{1}{N - \tau} \sum_{i=1}^{N-\tau} \Theta(\epsilon - ||x_i - x_{i+\tau}||_\infty) = \frac{1}{N - \tau} \sum_{i=1}^{N-\tau} R_{i,i+\tau}^{(e)}.$$  

The recurrence probability $P^{(e)}(\tau)$ can be viewed as a statistical measure of how often the phase angle associated with the system's trajectory (coherent or not) in the original phase space has increased by $2\pi$ within the time interval $\tau$. Analogously to periodic systems, it is possible to refer an increment of $2\pi$ to the phase angle, to a complex non-periodic trajectory $x(t)$ whenever $|x(t) - x(t + \tau)|_\infty < \epsilon$, or equivalently whenever $|x(t) - x(t + \tau)|_\infty < \epsilon$. If two systems are in PS, the phases of both systems increase by $k \cdot 2\pi$, with $k$ a natural number, within the same time interval $\tau$. Hence, looking at the coincidence of the positions of the maxima of $P^{(e)}(\tau)$ for both systems, it is possible to quantitatively identify PS by applying a two-step algorithm. First, we compute $P_{1,2}(\tau)$ for both systems. Second, the cross-correlation coefficient between $P_{1}(\tau)$ and $P_{2}(\tau)$ (Correlation between Probabilities of Recurrence) is evaluated using
\[ \text{CPR} = \frac{\langle \hat{P}_1(\tau) \hat{P}_2(\tau) \rangle}{\sigma_1 \sigma_2}, \]  
where \( \hat{P}_{1,2} \) denotes that the mean value has been subtracted and \( \sigma_1 \) and \( \sigma_2 \) are the standard deviations of \( P_1(\tau) \) and \( P_2(\tau) \), respectively. If both systems are in PS, the probabilities of recurrence are maximal at the same time and CPR \( \approx 1 \) [21].

4. Results

4.1. Preliminaries

We now compare the results of applying both methods to the same test situation. The time series data used are generated by two non-identical Rössler systems [22] named 1 and 2, mutually coupled through the corresponding \( x \) and \( y \) components

\[ \begin{align*}
\dot{x}_1 &= -\omega_1 y_1 z_1 + \eta(x_2 - x_1), \\
\dot{y}_1 &= -\omega_1 x_1 + \eta y_1 + \eta(y_2 - y_1), \\
\dot{z}_1 &= 0.1 + z_1(x_1 - 8.5),
\end{align*} \]  
(9)

\[ \begin{align*}
\dot{x}_2 &= -\omega_2 y_2 z_2 + \eta(x_1 - x_2), \\
\dot{y}_2 &= -\omega_2 x_2 + \eta y_2 + \eta(y_1 - y_2), \\
\dot{z}_2 &= 0.1 + z_2(x_2 - 8.5)
\end{align*} \]  
(10)

where \( \eta \) is the coupling strength, and \( \omega_{1,2} \) introduce a small mean frequency mismatch between the two oscillators. We set \( \omega_1 = 0.98, \omega_2 = 1.02 \), and use the Runge–Kutta 4th order method with integration time-step equal to 0.001. Parameter \( a \) governs the topology of the attractor. For \( a = 0.15 \), the trajectory of the uncoupled Rössler oscillator cycles around the unstable fixed point \( (x_0, y_0) \approx (0, 0) \) in the \((x, y)\) subspace. In this case, the oscillator has a phase-coherent dynamics so that the rotation angle about the unstable fixed point \( (0, 0) \)

\[ \phi = \arctan \frac{y}{x} \]  
(11)
can be defined as the phase (see Fig. 2). For \( a = 0.29 \) the trajectory no longer cycles continuously around \((x_0, y_0)\), and some \( \text{max}(y) < y_0 \) associated with faster return orbits occur. This is the so-called funnel (non-phase-coherent) regime where phase, as defined by Eq. (11), no longer applies [23]. However, as the trajectory has a single direction of rotation around some point that can be considered the center of rotation, the phase angle can be defined using [24]

\[ \phi_B = \arctan \frac{\dot{y}}{\dot{x}} \]  
(12)

Figs. 2(c) and 2(d) depict the Rössler attractor in \((x, \dot{y})\)-plane showing that in both phase-coherent and non-phase-coherent cases a center of rotation can be located, and a proper measure of phase angle \( \phi_B(t) \) can be made. Therefore, with the phase angle well-defined, the phase synchronous regime can be identified with the condition

\[ \Delta \phi(t) \equiv |n \phi_1(t) - m \phi_2(t)| \leq \text{const} \leq 2\pi, \]  
(13)

where \( n \) and \( m \) are integers. Figs. 3(a) and 3(b) show phase differences between systems 1 and 2 for \( a = 0.29 \) and coupling parameter \( \eta \) values equal to 0.09 and 0.1 in Figs. 3(c) and 3(d), respectively. In the first case, we notice epochs of PS indicated by the plateaus with phase slips of \( 2\pi \) between successive plateaus. No phase slips are detected in the second case showing a continuous plateau typical for PS states.

![Fig. 2](image-url)  
![Fig. 3](image-url)

We implement the LBFM method by driving two identical arrays of Poincaré oscillators with the signals \( X_{1,2}(t) \) whose synchronization relation we want to determine, and using equations

\[ \begin{align*}
\dot{\phi}_B &= \omega_B - \epsilon X_{1,2}(t) \sin \phi_B, \\
\Omega_B &= \lim_{t \to \infty} \left[ \phi_B(t) - \phi_B(0) \right]/t.
\end{align*} \]  
(14)

where the signals \( X_{1,2}(t) \) are normalized to have zero mean and unit variance. Our initial analysis consists of evaluating the sensitivity of the methods in detecting PS between two Rössler systems in non-phase-coherent regime with \( a = 0.29, \omega_1 = 0.98 \) and \( \omega_2 = 1.02, \) and \( \eta \) is varied in the range \([0.03, 3.0]\).

Fig. 4 shows the outcome of the LBFM method applied to data series with 50,000 points. We use an ensemble of 100 auxiliary oscillators with natural frequencies uniformly distributed in the interval \([0.6, 1.5]\). Fig. 4 shows the evolution of the results as the coupling strength increases. In cases (a) and (b), with weaker coupling, the plateaus do not coincide, as opposed to cases (c) and (d).
Fig. 4. Results of applying the LBFM method to a data series of 50,000 points to detect PS. The coupling strength between the systems is the following: (a) \( \eta = 0.03 \); (b) \( \eta = 0.09 \); (c) \( \eta = 0.1 \); (d) \( \eta = 0.2 \). Analyzing the coincidence among the plateaus it is possible to conclude that there are PS for cases (c) and (d).

Fig. 5. Results of applying LBFM with \( \omega \in [0.75, 0.95] \) to detect phase synchronism. The coupling strength between the systems are the following: (a) \( \eta = 0.03 \); (b) \( \eta = 0.09 \); (c) \( \eta = 0.1 \); (d) \( \eta = 0.2 \). Analyzing the coincidence among the plateaus it is possible to conclude that there are phase synchronism only for case (d).

Fig. 6. Synchronization transition result for LBFM method for different coupling values for the systems.

Fig. 7. Results of applying the PSMR method to a data series of 35,000 points to detect PS. The coupling strength between the systems is the following: (a) \( \eta = 0.03 \); (b) \( \eta = 0.09 \); (c) \( \eta = 0.1 \); (d) \( \eta = 0.2 \).

where stronger coupling clearly produces coincidence of plateaus. Note the consistency of these results with those depicted in Fig. 3. We can quantify the LBFM results by calculating the middle point \( \Omega_p \). If \( \Omega_p \) is well defined and coincident, we can claim that PS is taking place. In this case, from the figures we see the presence of more than one plateau for each system. The method's accuracy can be improved by selecting another ensemble of oscillators whose frequency are distributed around a specific plateau. In this zooming in, we apply LBFM using an ensemble of oscillators with natural frequencies \( \omega \in [0.75, 0.95] \). The results are shown in Fig. 5. From Fig. 5, we see that PS just sets in case (d), for which differences between plateaus, \( \Delta \Omega_p = \Omega_p^2 - \Omega_p^1 \), is \( \approx 0 \). Fig. 6 shows \( \Delta \Omega_p \) vs. \( \eta \). PS between the systems sets in when \( \eta > 0.15 \).

Fig. 7 displays the results obtained with the PSMR method applied to data series with 35,000 points. Notice how the peaks become more coincidental with increasing values of \( \eta \) and a corresponding increase in CPR values as well. Notice the ten fold increase in the CPR value occurring during the transition to PS between the two oscillators.

Both LBFM and PSMR methods clearly indicate the PS dependence on the coupling parameter \( \eta \), and present basically the same sensitivity. However the PSMR provides a quantifier for measuring PS-CPR-making it possible to characterize the effect of increasing the coupling strength for regions where PS does not set in, as shown in Fig. 8.

4.2. Data size effects

Now, we analyze the effect of the length of the time series on the performance of the methods. The system parameters are \( a = 0.29, \omega_1 = 0.98 \) and \( \omega_2 = 1.02 \), which means that the two non-identical Rössler systems are in non-phase-coherent regime. The coupling strength used for all cases is \( \eta = 0.2 \), which implies that the systems are in PS. Fig. 9 presents the results of applying the LBFM to data series with number of points \( N \) equal to 5000, 10,000, 25,000 and 35,000. As the number of points increases, the plateaus become more well defined and coincident. For 5000 points, the shape of the curves does not have the expected appearance with well defined plateaus, so that the result is
Fig. 8. The PSMR is sensitive enough to detect a transition toward PS as the coupling strength parameter between the systems increases.

Fig. 9. Results of applying LBFM method for identifying PS to data series of different number of points, which are the following: (a) 5000; (b) 10,000; (c) 25,000; (d) 35,000.

Fig. 10. Performance result for the LBFM. This method produces good and consistent results for a data series greater than 25,000 points.

Fig. 11. (a) Result of applying PSMR method for identifying PS to data series of 5000 points; (b) Changing the scale to show details of the result.

Fig. 12. (a) Result of applying PSMR method for identifying PS to data series of 10,000 points; (b) Changing the scale to show details of the result.

Fig. 13. Performance result for the LBFM. This method produces good consistent results even for a data series with a small number of points.
inconclusive. For 25,000 points, the plateaus start to become more visible. However, 35,000 points seems to be the minimum number of points beyond which the curves start to display the feature that is expected for systems in PS. In Fig. 10, we depict the method’s performance as a function of the number of points in the data series. We just put in the graphic results for situation in which the plateaus are visible and so can be taken in consideration regarding the LBFM method. It is possible to see that after a threshold, the graphic settles down to a constant level. It means that the LBFM result is consistent, indicating PS.

For the PSMR method, as can be seen from Fig. 11, with just $N = 5000$ it is possible to be sure that there is a PS. In Fig. 12 we can see that by using a larger data series the confidence in the diagnosis of PS provided by the method is strengthened.

Although these results depend on the system dynamics, we can say that PS can be identified by the PSMR method with just a considerably small data series, which is not the case for the LBFM method. Fig. 13 shows how the parameter CPR changes as the number of points in the data series increases. We can see that just 1000 points are enough to produce a good result with the method. This capability of detecting PS with a reduced number of points allows the use of PSMR method even for data series in which the coupling parameter presents small changes or fluctuations during the data collecting time. This is not the case for LBFM, that is only applicable to data series for which the coupling between the systems is invariant over the data collecting time.

4.3. Noise effects

Next, we investigate the effect of adding noise to each component of the system. The signals become $\tilde{x}_{1,2}(t) = x_{1,2}(t) + \sigma \theta(t)$ (the same for the $y_{1,2}$ and $z_{1,2}$ components), where $\theta(t)$ is an independent uniformly distributed noise with zero mean and standard deviation $1$. The system parameters are $a = 0.29$, $\omega_1 = 0.98$ and $\omega_2 = 1.02$, which means that the two Rössler systems are in the non-phase-coherent regime. The coupling strength used for all cases is $\eta = 0.2$, so that, in the absence of noise, the two oscillators are in PS.

Fig. 14 shows the results for the LBFM method for data series with 30,000 data points, and $\sigma$ equals to, respectively, 0.0, 0.2, 0.5
and 0.7. It clearly indicates that it is very robust to the presence of even high intensity level of additive noise. The same noise level produces a result for the PSMR method, as depicted in Fig. 15. The increase of the intensity level of the additive noise implies in a CPR decrease. Actually, PSMR is very sensitive to the presence of noise, as can be seen from Fig. 16. It happens because the effect of noise impose bias to the system trajectories so that they present episodes in which they deviate from each other a distance that goes beyond the threshold $\epsilon$ in Eq. (6).

4.4. Results from experimental data

In order to illustrate the applicability of both LBFM and PSMR methods to experimental data we present results of PS detection on a plasma system forced with a low amplitude sinusoidal drive. The experimental setup and other details can be found in Ref. [13]. We start with a signal output from a plasma discharge tube subject to a voltage of 850 volts. The power spectrum of the signal is broad with a predominant frequency of 6960 Hz and with the largest Lyapunov exponent positive, indicating the plasma's chaotic character. In Fig. 17 we show the reconstructed attractor obtained from the measured data series.

A signal driven by a sinusoidal function with a 0.2 volts amplitude, is analyzed using both LBFM and PSMR methods, and the results shown in Fig. 18. For the LBFM method, we see the plateau that is associated with the sine wave with frequency equal to 6960 Hz, but there is no plateau related to the data series from the experiment. It indicates that there is no PS at all. For the PSMR (left graph), the CPR value indicates no synchronization, but we can see a small effect of the sine wave on the attractor. It is unveiled by the coincidences between the maxima of their respective probabilities. Nevertheless, the maximum associated with the data series is very small and broader $\hat{P}(\epsilon)(\tau)$. It suggests a very weak effect of PS.

This situation changes dramatically if the amplitude of the sine wave is increased to 0.4 V, as can be seen in Fig. 19. In this case both methods clearly show PS between the plasma and the sinusoidal drive. Note also that the outcome from the PSMR method...
yields more information about the ongoing process. Besides indicating the presence of the synchronous phenomenon, it also allows its quantification through the CPR index, in addition to helping understand the development of the phenomenon.

5. Concluding remarks

PS in chaotic systems has been identified in many natural and experimental processes. It is considered ubiquitous in nature and a fundamental mechanism mediating many natural phenomena. Its identification can be difficult especially when the signals involved come from chaotic systems in non-phase-coherent regimes. Recently, two methods were proposed for identifying PS between chaotic systems: The Locking-Based Frequency Method (LBFM) and the Phase Synchronization by Means of Recurrence (PSMR). In this study we compare the performance of both methods involving two coupled non-identical Rössler oscillators, as well as an experimental periodically driven chaotic plasma discharge. We draw the following conclusions:

• Both methods are equally computational intensive;
• Both methods are relatively easy to implement;
• Both methods give a good resolution in identifying PS;
• PSMR is sensitive enough to identify tendencies associated with a varying parameter as changes in this parameter value move the system from no synchronization to PS;
• Besides the detection of PS, PSMR also allows its quantification through the CPR index. It also provides a more complete picture of the development of the PS process;
• Both systems are applicable to situations where the systems are far away from the point where the transition to PS happens;
• In some cases, LBFM requires larger data series;
• PSMR seems to be reliable in cases where the coupling parameter undergoes small changes or fluctuations during data acquisition;
• PSMR is much more sensitive to the presence of additive noise than LBFM.

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References