Quantifying Stickiness in 2D Area-Preserving Maps by Means of Recurrence Plots

by

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Dedicated to Radah and George.
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Stickiness is a ubiquitous property of dynamical systems. However, recognizing whether an orbit is temporarily ‘stuck’ (and therefore very nearly quasiperiodic) is hard to detect. Outlined in this thesis is an approach to quantifying stickiness in area-preserving maps based on a tool called recurrence plots that is not very commonly used. With the analyses presented herein it is shown that recurrence plot methods can give very close estimates to stickiness exponents that were previously calculated using Poincaré recurrence and other methods. To capture the dynamics, RP methods require shorter data series than more conventional methods and are able to represent a more-global analysis of recurrence. A description of stickiness of the standard map for a wide array of parameter strengths is presented and a start at analyzing the standard nontwist map is presented.
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Chapter 1

Introduction

Stickiness is a ubiquitous property in physics. From general relativity to chaos theory to simple random walk problems, trajectories can appear chaotic or random on large time scales but resemble quasiperiodic orbits very closely for smaller time scales. It is an important property to understand but it has not been investigated very thoroughly to date. Presented herein is an attempt at quantifying stickiness in two Hamiltonian maps that represent a large sampling of physical phenomenon.

Stickiness is going to be quantified by means of recurrence plots, a data analysis technique introduced in 1987 by Eckmann et al. [9] that is based off of the Poincaré recurrence theorem. This theorem states that, for a given area-preserving, bounded phase space, any point will, after continued mapping, have recurrences that come arbitrarily close to the initial point after a sufficiently long time. Recurrence plots use the recurrence information in a data series to produce a 2D plot that compares every point $x_i$ to every other point $x_j$ for natural numbers $i$ and $j$ representing discretized time. For a data series of length $N$, a recurrence plot is $N \times N$ in area and contains $N(N-1)/2$ unique recurrence comparisons, a feature that allows recurrence plots to make predictions on smaller data sets than would be required by, for instance, Lyapunov exponents. The texture of the plots contains information about events such as period bifurcation, chaos transitions, periodicities, or other important
properties in the data series. Data from two different Hamiltonian systems will be analyzed in order to measure stickiness for different parameter strengths. A great deal is known about the analytic structure of these systems but statistical properties such as stickiness have not been sufficiently studied. The chapters contained herein outline a method that measures the ‘stickiness’ of chaotic trajectories in the standard map and the standard nontwist map, a method that is extensible to other data series that one would want to probe, such as systems that possess significant transport barriers.

In Ch. 2 two 2D area-preserving maps will be introduced, including a discussion of their properties that help one understand the mechanisms of stickiness. In Ch. 3 stickiness will be defined more thoroughly with a motivation from the Poincaré recurrence theorem as well as examples of stickiness in other fields. Ch. 4 is an overview of recurrence plots in general and recurrence rate in particular, the diagnostic used to quantify stickiness in this thesis, in particular. Canonical examples show the utility of the methods. Ch. 5 is a walk-through of the methods used to quantify stickiness for the maps introduced previously. The results of these methods are given and discussed in Ch. 6 with speculation and further research directions given in Ch. 7.
Chapter 2

2D Area-Preserving Maps

Stickiness is analyzed for two important area-preserving maps. The Chirikov standard map (SM) is defined on a torus, with the phase space coordinates \((x, y) \in [0, 1) \times [0, 1)\) and parameter \(k \in \mathbb{R}^+\) (Ref. [1]),

\[
\begin{align*}
M_{SM}: \quad y' &= y - \frac{k}{2\pi} \sin (2\pi x) \\
x' &= x + y' \quad \text{mod 1}
\end{align*}
\] (2.1)

and the standard nontwist map (SNM) (Refs. [6] and [13]):

\[
\begin{align*}
M_{SNM}: \quad y' &= y - b \sin (2\pi x) \\
x' &= x + a \left[1 - (y')^2\right] \quad \text{mod 1}
\end{align*}
\] (2.2)

with phase space defined on a cylinder with coordinates \((x, y) \in [-.5, .5) \times \mathbb{R}\). In both cases above \(x\) represents a position coordinate and \(y\) represents momentum.

These maps are significant because of their wide applicability in physical descriptions. Each will be treated individually in Secs. 2.2 and 2.3 but first some commonalities will be defined in Sec. 2.1.

2.1 Properties of Area-Preserving Maps

Area-preserving maps, as their name suggests, have a phase space volume that is constant in time. This can be verified by computing the determin-
nant of the Jacobian for the map:

\[ L = DM = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}. \] (2.3)

For maps, 2.1 and 2.2 it can be easily verified that

\[ \det L = 1. \] (2.4)

From the matrix \( L \) one can obtain the eigenvalues \( \lambda_{1,2} \) which will help in classifying the points of the map. The characteristic equation of \( L \) takes the form:

\[ \lambda^2 - Tr(L) + \det L = \lambda^2 - Tr(L) + 1, \] (2.5)

where Eq. (2.4) was used, and \( Tr(L) \) is the sum of diagonal elements of \( L \). The eigenvalues are then given by

\[ \lambda_{1,2} = \frac{1}{2} \left( TrL \pm \sqrt{(Tr(L))^2 - 4} \right). \] (2.6)

### 2.1.1 Winding Number

An important measure for area-preserving maps is the *winding number* \( \omega \), a number that describes the time average of \( n \) iterates of a map \( M \) on \( x \) as \( n \) tends to infinity. It is given by the following formula (where \( \hat{x}_n = x_n \) sans modulus)[13]:

\[ \omega = \lim_{n \to \infty} \frac{\hat{x}_n}{n}. \] (2.7)

Seen another way, the winding number is the ratio of the “short way speed” \( m \) divided by the “long way speed” \( n \) around a torus [21]. As an example, take \( k = 0 \) for the standard map lifted off the torus (i.e., \( (x, y) \in \mathbb{R} \times \mathbb{R} \)). Then,

\[ y' = y \]

\[ x' = x + y' \] (2.8)
which means that $y_0 = y_1 = ... = y_n$ for $n \in \mathbb{Z}$. For the $x$-coordinate, every iteration adds another $y_0$. Therefore,

$$x_n = x_0 + ny_0, \quad (2.9)$$

which is just the kinematic equation $[x_n - x_0]/n = y_0$ with constant velocity $y_0$, time $n$ and position $x$ (no applied forces since $k = 0$). Using Eq. (2.7) on Eq. (2.9) gives

$$\omega = \lim_{n \to \infty} \frac{x_0 + ny_0}{n} = y_0. \quad (2.10)$$

### 2.1.2 Orbits

Orbits in area-preserving maps can be of three main types: periodic, quasiperiodic, and chaotic. The first two orbit types have well-defined winding numbers while chaotic orbits do not.

**Periodic orbits** are trajectories composed of a finite set of points such that, for integers $p$ and $q$, after $p$ actions of $M$ on phase space points $(x, y)$ they are mapped to $(x + q, y + p)$, where there are $q$ wrappings around the torus the short way and $p$ wrappings around the long way which brings the points back to their original position $(x, y)$. Periodic orbits can be further classified as elliptic, parabolic, or hyperbolic. This classification can be determined either by evaluating the eigenvalues [Eq. (2.6)] or by using Greene’s residue on the Jacobian [Eq. (2.3)] given above:

$$R = \frac{1}{4} (2 - Tr (L)), \quad (2.11)$$

where $L$ is evaluated at the phase space points $(x_p, y_p)$ and parameter value $k$ where the period-$p$ periodic orbit occurs.

Periodic orbits fall into three main types ([21] and [13]):
1. **Hyperbolic** orbits occur when $R > 1$ ($\lambda = \lambda_1 = 1/\lambda_2 \in \mathbb{R}^-$) or $R < 0$ ($\lambda = \lambda_1 = 1/\lambda_2 \in \mathbb{R}^+$) with $|\lambda| > 1$. Here, since the magnitude of one of the pairs of eigenvalues is larger than one, repeated mappings of a small perturbation $\delta z'$ away from the periodic orbit gives an unstable manifold (expanding). Likewise, repeated mappings of a perturbation with $\lambda_2$ gives a stable manifold (contracting). Therefore, hyperbolic orbits contain both a stable and unstable manifold.

2. **Elliptic** orbits occur when $0 < R < 1$ which corresponds to $\lambda = \lambda_1 = \lambda_1^* \in \mathbb{C}$. Complex eigenvalues correspond to small perturbations $\delta z'$ away from periodic orbits that encircle the fixed point. The magnitude of both the eigenvalues is less than one, implying stability.

3. **Parabolic** orbits occur for $R = 0$ or $R = 1$, which corresponds to $|\lambda| = 1$. Parabolic orbits are the limiting case between elliptic and hyperbolic orbits.

**Quasiperiodic orbits** are invariant sets that fill a curve densely, a property of their irrational winding number. Since continued action of $M$ on phase space coordinates will never bring any point exactly back to an earlier point, quasiperiodic orbits cannot be characterized by integer $p$ and $q$ as was the case for periodic orbits. Just as the density of rational numbers on the real line $[0, 1)$ is infinitesimally small, so too is the density of periodic orbits to other orbit types in non-linear maps. Orbits with rational $\omega$ consist of a finite number of points while those with irrational $\omega$ densely fill a curve in phase space. Quasiperiodic orbits have the property of acting as transport barriers between stochastic layers since the set densely fills the curve on which $M$ maps points in the trajectory. Quasiperiodic orbits can also be referred to as **invariant tori**,
a reference to the curve forming an invariant set under continued mapping of $M$, while \textit{tori} refers to the curve being a one-dimensional torus that cannot be contracted to a point since it spans the whole $x$ domain, as opposed to the case for the curves that encircle elliptic orbits. These curves are also called KAM tori after \textit{KAM theory}, which describes “the persistence of invariant tori under small perturbations from integrability”\cite{13}. The most persistent of these tori is the one with the “most irrational” winding number, given by the golden mean. Represented in different forms, $\omega_\gamma = \frac{1}{\phi} = (\sqrt{5} - 1)/2 = [0; 1, 1, 1, ...]$, where the notation

$$[a_0; a_1, a_2, ...] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + ...}}$$

refers to a continued fraction expansion \cite{16}. This torus, also called the \textit{golden circle} or \textit{critical torus}, is fractal in nature, a property that plays a role in the stickiness of adjacent chaotic orbits. A critical torus is easily seen snaking across the whole $x$ domain of the phase space in the SNM plot [Fig. 2.2(b)] with $(a, b) = (0.686049..., 0.742494...)$.

Likewise, in the SM at $k \approx 0.971635$, the blue curve at the top of the bare region above the period two orbits has a winding number that closely approximates the golden torus.

\textit{Chaotic orbits} are paths that have a characteristic rate of divergence $\tau_\beta$ defined as follows:

$$\delta z_n = \delta z_0 e^{\frac{\omega}{\beta}}$$

for time $n$. This equation says that points initially separated by a small amount $\delta z_n$ will diverge at an exponential rate given by $\sigma$. Chaotic orbits are the orbits that densely fill the space between quasiperiodic orbits as the map becomes more nonlinear (i.e., larger $k$ or $(a, b)$). Chaotic orbits are the areas of the
maps in Figs. 2.1 and 2.2 that have a homogeneous color distributions. All maps except for $k = 0$ on the SM display chaotic regions.

### 2.1.3 Island Chains

Island chains, also referred to as invariant sets or cantori because of their similarities to Cantor sets, form around elliptic periodic orbits in area-preserving maps. By the Poincaré-Birkhoff theorem, island chains come in multiples of two, with an elliptic point always complimented by a hyperbolic one. These chains are easily visible in the Figs. 2.1 and 2.2. For the SM, a 1-period elliptic orbit occurs at $(x, y) = (.5, 0)$ and its hyperbolic compliment is at $(0, 0)$; a 2-period orbit has elliptic fixed points at $(0, .5)$ and $(.5, .5)$ with the hyperbolic fixed points sandwiched between in the chaotic regions. Island chains have the important property of acting as semi-permeable transport barriers, another factor in stickiness (Ref. [19]).

### 2.2 Standard Map

The SM can describe situations as diverse as particle dynamics in accelerators, comet dynamics in the solar system, and charged particle confinement in magnetic mirrors, among other phenomena [3]. The SM comes from the Hamiltonian

$$H(p, q, t) = \frac{p^2}{2m} - \frac{k}{2\pi} \cos (2\pi q) \delta(t - n\tau)$$

with canonical variables $p$ and $q$, parameter strength $k$, and $\delta(t - n\tau)$ represents a $\delta$-function impulse at time $n\tau$ for any integer $n$ and arbitrary $\tau$. Physically, it is a ‘kicked rotor’ in the absence of gravity and friction.

Figure 2.1 displays the Poincaré sections of the map [Eq. (2.1)] at different values of $k$. As $k$ grows from zero, resonant tori (i.e., those with a rational
winding number for \( k = 0 \) are quickly destroyed and become enveloped by nested KAM curves (see [22] and [15]). At the center of the nested curves are elliptic fixed points that have periodic orbits. The more irrational the rotation number the more staying power the invariant tori possesses. The last remaining invariant torus breaks at a critical value called *Greene’s number*: \( k_g \approx 0.971635406 \) (see Fig. 2.1(b)). As \( k \) grows larger, the area bounded by KAM curves (those that encircle elliptic fixed points) becomes smaller and smaller as chaotic regions fill the phase space (Fig. 3.1). At \( k = 4 \), the elliptic fixed point at \((0, .5)\) experiences a period doubling bifurcation into two new elliptic orbits, breaking a significant transport boundary. These bifurcations are happening at all levels of the map for different \( k \) values because it is self-similar. This self-similarity becomes evident in the scaling of the distribution of stickiness durations in Sec. 6.1.

### 2.3 Standard Nontwist Map

The standard nontwist map (SNM) is another Hamiltonian map that shows up in a variety of places. Notable among these is a description of magnetic field lines in toroidal plasma devices such as tokomaks and stellerators [7]. Figure 2.2 contains four Poincaré sections of the SNM. Plots (a) and (c) are below criticality, (b) is at criticality, and (d) is well above criticality.
Figure 2.1: The standard map with 70 trajectories (each a different color) with initial conditions sampled randomly
Figure 2.2: The standard nontwist map with 40 trajectories (each a different color) with initial conditions sampled randomly
Chapter 3

Stickiness

To define stickiness one first has to understand the concept of recurrence. In dynamical systems, recurrence took a mature form under the Poincaré recurrence theorem. The theorem goes as follows [22]: for some time-independent Hamiltonian $H = H(p, q)$ that has bounded orbits (that is, there are no energies $E = H(p, q)$ for $|p|$ or $|q|$ tending to $\infty$) there is a finite time interval during which a future point in the trajectory will occur arbitrarily close to the initial point. If one takes any initial point $z_0$ in phase space in a region $R_0$ with radius $\varepsilon$, the Poincaré recurrence theorem states that after a finite time $n$ there will be a point $z_n = M^n z_0$ that lies in the region $R_0$ regardless of the size of $\varepsilon$. Since Liouville’s theorem guarantees the conservation of phase space volume in Hamiltonian systems, the volume of any subset of points must also be conserved upon being mapped. Therefore, $\text{vol} (R_0) = \text{vol} (R_1) = \ldots = \text{vol} (R_p)$ for some integer $p$. Given that the total phase space volume is bounded, the sum of non-overlapping regions $R_i$ for any integer $i$ cannot be larger than the volume of the phase space. Therefore, some of the regions must overlap. Let $r$ and $s$ be integers such that $r > s$. Then for some $R_s$ and $R_r$ such that $R_s \cap R_r \neq \emptyset$. Applying the inverse mapping to both $R_s$ and $R_r$ gives $R_{s-1}$ and $R_{r-1}$ which must overlap as well. Applying the inverse mapping $s$ times to $R_r$ gives a region $R_{r-s}$ such that $R_{r-s}$ overlaps with $R_0$. 

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Stickiness, then, is a sort of meta-recurrence – a recurrence of quasiperiodic orbits. It is a property that shows up throughout the natural world but was not widely viewed as important until the discovery of chaos. Following Poincaré’s recurrence theorem, Boltzman calculated the average time for a particle to return to a position and velocity within certain bounds and found times that are super-astronomically large [23]. Because of this and arguments that the natural time scale of a system does not have a relation to its recurrence properties, recurrence analysis fell out of favor. It was with the discovery of chaos that the importance of recurrence was revived. In dynamical systems, stickiness is the tendency for a trajectory to contain an orbit that is ‘stuck’ with almost regular dynamics for time spans short compared to the overall chaotic orbit. A tangible analogous example is an asteroid entering into a near-stable orbit around a planet with a moon. Over the long term, the most likely scenario is that the asteroid would be ejected from the system or crash into one of the gravitating bodies. But for the time the asteroid was in its regular-like orbit, it can be thought of as being in the ‘sticky’ segment.

The stickiness properties of the SM are still not known with much definiteness. Attempts to calculate it have been made ([5], [12]). Many twist maps have been explored [23] but nontwist maps have not been treated as thoroughly (see Refs in [4]). Some basic statistical properties of Hamiltonian maps such as the standard (twist) map are yet unexplored [4].

3.1 Examples of Stickiness

As stated before, stickiness is a common phenomenon. Below are some examples of systems that exhibit stickiness in trajectories.

- Area-preserving maps (see Ch. 6)
• General Relativity – apparent in correlation functions that display algebraic power laws and in quasiperiodic and quasiperiodic-like orbits in solutions to the Mixmaster universe model (see Ref. [8])

• Weierstrass random walk – describes random walks that show self-similar behavior similar to the fractal-time errors found in network transmission ([23] and [14])

3.2 Quantification of Stickiness

![Figure 3.1: Fraction of phase space: bounded by the outermost invariant tori surrounding elliptic orbits (black), motion that is stochastic (orange), or motion that is ‘stable’ (blue). All measures are a function of the parameter $k$ in the standard map. Orange and blue data come from Ref. [1].](image)

It has been shown that the probability for a sticky event of length $\tau \gg 1$
in Hamiltonian twist maps obey statistics that are dependent upon properties of the phase space such as non-uniformity versus uniform mixing, ‘net traps’, fractal boundaries, ‘single marginal fixed points’ (Refs. [4] and [23]), presence of cantori, and islands-around-islands (Ref. [12]). In the case of the standard map, the non-uniformity of the phase space is determined by the strength of the parameter $k$. By introducing a measure $A_{IT}$ that represents the area of phase space bounded by the outermost invariant tori around elliptic orbits, one may be able to gain some insight into the statistics that govern the stickiness of the system (here we consider phase spaces in a unit square). $A_{IT}$ is plotted in Fig. 3.1. Similar measures were introduced in Refs. [1] and [20]. In the former, the percentage of phase space that has ‘stable’ behavior (meaning quasiperiodic orbits) is given by $\sigma_s$, while $A_{IT}$ does not take into account transport boundaries that span the phase space such as the critical torus. As can be seen from Fig. 3.1, a rough data series based on counting boxes that contain islands of stability, $A_{IT} = 0$ for $k = 0$ and peaks somewhere near the destruction of the critical torus at $k_g$ and then settles back towards 0 for $k \approx 2\pi$ or greater. The measure $A_{IT}$ coincides with the measure $\sigma_s$ introduced in Ref. [1] once the critical torus breaks, and the numbers line up within a reasonable error. It is believed by this author that this measure takes on a fractal-like structure if the map is analyzed with fine enough detail, similar to what was found for the Hénon area-preserving map in Ref. [20]. As will be discussed later, there are values of $k$ where the standard map is believed to be very nearly ergodic (e.g., $k \approx \frac{8\pi}{9}$, see [1]), but stable islands can quickly come into view again. For example, there is a small but noticeable upswing in $\sigma_s$ at $k = 6.59$ in Fig. 3.1.
3.3 Stickiness Statistics

As will be shown in Ch. 6, sticky duration events obey statistics that are dependent on the parameter strengths of the maps. As is commonly known, ergodic processes or those that are uniformly mixed exhibit an exponential decay:

\[
\rho(t > \tau) \sim e^{-\tau/\tau_\beta}.
\]  

(3.1)

Here \( \tau_\beta \) represents the inverse characteristic time of divergence of two close trajectories [seen in Eq. (2.13)]. For phase spaces that have significant transport barriers, such as those in Fig. 2.1 with \( k = k_g \) and \( k = 1.5 \), sticky durations have been found to obey an algebraic power law [23]:

\[
\rho(t > \tau) \sim \tau^{-\gamma}.
\]  

(3.2)

For the algebraic decay cumulative distribution function above, the greater the value of \( \gamma \), the faster the function decays, therefore the smaller the likelihood of long sticky events. Meanwhile, a small \( \gamma \) gives a flatter decay and a higher likelihood for longer sticky events. Fig. 3.1 gives a generic picture of the overall number of transport boundaries in the standard map and in this way may act as a predictor for the statistics of sticky events. For instance, one might expect there to be a relatively small \( \gamma \) (more sticky events) at \( k_g \) since the graph of \( A_{IT} \) peaks there and a very high \( \gamma \) or an exponential distribution at \( k \approx 8\frac{8}{9} \) since the map is thought to be close to ergodic there. Factors that Fig. 3.1 does not encode are the fractal properties of the curves or the permeability of the island chains. As will be found in Ch. 6, the plot has some predictive power despite some obvious flaws.
Chapter 4

Recurrence Plots

Realizing the limits of some of the methods used to calculated dynamical quantities such as information dimension, entropy, Lyapunov exponents, and dimension spectra, Eckmann et al. [9] introduced a diagnostic called recurrence plots (RP) that makes use of recurrences to encode information about dynamical systems. In Sec. 4.1, RPs are defined and the preceding sections discuss the elements of RPs and its use in dynamical systems.

4.1 Definition of Recurrence Plots

RPs are two-dimensional plots defined as

\[ R_{i,j} \equiv \Theta (\varepsilon - \| \vec{x}_i - \vec{x}_j \|) \]  

(4.1)

for a discretized data series \( \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_N\} \) of length \( N \) in an \( M \)-dimensional space. The use of the Heaviside-Theta function limits the components of standard RPs to ones and zeroes. As such RPs are a kind of contour plot with only two levels: on a recurrence or not. See Fig. 4.1 for a small sampling of RPs alongside the system from which they are made.

There is a wide variety of recurrence plots (some thresholded, some not) but the present analysis uses only the most basic and common type. Ref. [11] contains a thorough overview of RP types and their uses. RPs are used widely throughout the sciences although the use in the respective disciplines is fairly
limited. One of the strengths of the tool is its ability to capture dynamics in short time series, which, in most cases, is several magnitudes lower than what is required to calculate, for instance, Lyapunov exponents[12].

4.2 Elements of Recurrence Plots

4.2.1 Norm

The general definition for the norm of a vector is given by

$$\|\vec{x}\|_p \equiv \left( \sum_{i=1}^{M} |x_i|^p \right)^{\frac{1}{p}} \quad (4.2)$$

for some $M$-dimensional vector $\vec{x}$, natural number $p$, and absolute value $| \cdot |$. The Euclidean norm comes from $p = 2$ and the ‘taxi cab’ norm comes from $p = 1$. The special case of $p \to \infty$ is known as the max norm. It is least costly computationally since it only chooses the largest component of the vector in the summation – as $p$ tends to $\infty$ the largest component grows so large that the other terms in the summation become insignificantly small in comparison when taken to the $p^{th}$ power. That is,

$$\|\vec{x}\|_\infty = \max \{|x_i|\}. \quad (4.3)$$

4.2.2 Recurrence Threshold $\varepsilon$

A recurrence is defined as the normed difference between two points that is less than a chosen threshold parameter. This parameter, $\varepsilon$, to a large part controls the constitution of the 2D plot. A large $\varepsilon$ results in a high number of recurrences and washed-out statistics; a small $\varepsilon$ results in a relatively small number of recurrences and precise but potentially meaningless statistics. See
Fig. 4.2 for a sample of how the density of a recurrence plot (the number of recurrences divided by the length of the data series to the second power). There is no solid method for choosing an ideal parameter but a good rule of thumb is that $\varepsilon$ “should be not exceed 10% of the mean or the maximum phase space diameter”[11]. The characteristics of the data series ultimately decide the value of $\varepsilon$ one should choose. For instance, if the data is fairly stationary the recurrence plot will be saturated for all but a very small $\varepsilon$. The method used in this paper takes $\varepsilon$ as equal to a percentage of the standard deviation of a known data series with which significant properties in the data series under consideration mirrors at some points in its trajectory. In practice, there is a sizable window for choosing a threshold value that still admits reasonable statistics. Fig. 4.2 is a sample of how the density of an RP changes for different segments of the overall trajectory as the threshold value changes using data from the standard map. As can be seen for the case where $\varepsilon = .01$ (the bottom-most blue line) significant features of the line lie very close to the average behavior. Likewise, when $\varepsilon = .21$, the data fluctuates greatly and the significant features at about $\tau = 28$ and $70 \leq \tau \leq 80$ appear washed-out compared to the other data in the line. The values of $\varepsilon$ between .06 and .15 seem to offer a good compromise. To date there has not been a solid analytical method for choosing $\varepsilon$ for any data series.

4.2.3 Embedding Dimension and Delay Coordinates

Not all like-points are recurrences. For instance, in a sine wave $x(t) = \sin t$, two points of a full wavelength have the same numerical coordinate on an amplitude versus time graph, but slopes of the opposite sign. Therefore, these two points are in a different state in the phase space. Embedding is the process
of re-expressing the data into a higher dimension so that unlike phase space points no longer coincide at the same point. The process effectively untangles a data set. By doing this, one rids the data of false recurrences such as the case for the sine wave. The new coordinates, called delay coordinates, come from taking points $x(t)$ and forming coordinates in some new dimension, $m$:

$$z(t) = (x(t), x(t - T), x(t - 2T), ..., x(t - (m - 1)T)),$$  \hspace{1cm} (4.4)

for an $m$-dimensional vector $z(t)$, where

$$m \geq 2d + 1$$  \hspace{1cm} (4.5)

is generally adequate to untangle a system that is $d$-dimensional. In the case of the sine wave above, $m = 2$ is sufficient to uniquely identify the state of each point. This result, Eq. (4.5), is called Taken’s embedding theorem. More can be found at Refs. [10] and [22].

### 4.3 Interpretation of Recurrence Plots

Recurrence plots encode information about the recurrence properties of a given data series. Using some canonical examples one can develop an intuition of the meaning of the structures on the map. As can be seen by the norm $\| \cdot \|$, recurrence plots are symmetric about a line of identity (LOI) that corresponds to the obvious subthreshold values $\| x_i - x_j \|$ when $i = j$. If a mirror were placed along the LOI the image seen would be identical to the portion of the RP that the mirror is blocking. This is a property of the norm being symmetric with switching $i$ and $j$.

Examples of recurrence plots for familiar systems are presented in Figs. 4.1. In Fig. 4.1(a), one can see that various diagonal structures of varying separation characterize quasiperiodic motion. Measuring the vertical distance
between any two lines gives the period for a recurrence, the return time. Quasiperiodic orbits are characterized by three different return times [12]. In periodic systems, RPs contain a series of evenly separated diagonal lines from which the period, the only return time for this type of orbit, can easily be discerned. Stochastic orbits, such as the one in Figs. 4.1(c) and (d) of the logistic map [see Eq. (4.8)], are characterized by fleeting recurrences that show up as single, unconnected points in the RP, but portions of the trajectory that contain regular motion are easily visible as the short diagonals along the LOI and the collection of points around time \( t = 280 \). Another important feature are straight horizontal or vertical lines (the same under the symmetry of RPs). These features are the result of motion that is laminar or constant in time. This occurs in the trajectory in Fig. 4.1(c) around time \( t = 270 \) where the population value is near zero for 5 to 10 units of time. In Figs. 4.1(e) and (f), one can see the recurrence properties of the Lorenz attractor in phase space and as a recurrence plot. The initial point (red dot) starts in the center front of Fig. 4.1, and can be seen as starting far enough away from subsequent points that it is outside the recurrence radius until time \( t \approx 1 \) second. At this point, the map seems periodic but possesses something similar to drift until this behavior recurs around about time \( t \approx 12 \) seconds. As the system drifts more and more, the various parts of the trajectory become further and further apart on average, explaining the rather high density of recurrence points in the lower left quadrant and shorter diagonal segments in the upper right quadrant.
4.4 Recurrence Quantification Analysis

Recurrence quantification analysis (RQA) is the analysis that turns the qualitative, topological properties of an RP into quantitative measures. These measures are referred to as *ad hoc* because they are not in general measuring a common quantity directly, but instead they act as tools to elucidate important quantities.

Perhaps the easiest measure one can construct based on RPs is the density of recurrences in a plot. This would be obtained just as any other density would:

\[
RR(\varepsilon) = \frac{1}{N^2} \sum_{i,j=1}^{N} R_{i,j}(\varepsilon),
\]

for a data series of length \(N\). Recurrence rate, like the other measures, is sensitive to the choice of a threshold value for recurrence, but there is a wide range of \(\varepsilon\) where the significant features of the RR are not washed out in noise or constrained by too small a value of \(\varepsilon\).

Other important measures are:

1. **DET** – a measure of the diagonal lines in a given recurrence plot that is based on the histogram of diagonal line lengths;
2. **L** – the average diagonal line length;
3. **LAM** – a measure of the horizontal/vertical lines in a recurrence plot based on a histogram of horizontal and vertical line lengths;
4. **TT** – the average length of vertical/horizontal structures;
5. **ENTR** – which relates the probability of finding a diagonal line of a specific length to the Shannon entropy.
4.5 Windowed Recurrence Plots

As they stand, the RQA measures are non-dynamic. Instead they represent a measure of the global properties of the system, a potentially meaningless measure for long data series where short-lived but significant behavior may be dominated by a more global behavior. Specifically, chaotic orbits may contain sections of the trajectory that very closely resemble quasiperiodic and periodic orbits. As a way to show the dynamical behavior of the trajectory one can take RQA measures of small windows of the overall data series in such a way that steps through the series along the LOI of a recurrence plot of the full data series. In this way the RQA measures become time-dependent and significant events in the data series such as period bifurcation, chaos transitions, or sticky events become evident. Figs. 4.2 and 4.3 are examples of this. When taken in windows of length $w$ and with steps of $\tau$, Eq. (4.6) becomes

$$RR(\varepsilon, \tau, w) = \frac{1}{w^2} \sum_{i,j=1+\tau}^{w+\tau} R_{i,j}(\varepsilon).$$

(4.7)

This method of windowing can be easily extended to the other measures as well, but each measure is sensitive to different data series depending on the circumstance and use of the RQA measure (see, e.g., [11] and [12]).

An example of how recurrence quantification analysis can elucidate features in data is Fig. 4.3. The black plot is of the logistic map

$$x_{n+1} = rx_n (1 - x_n),$$

(4.8)

an iterated 1D map used to model population dynamics among other things. The red plot is the recurrence rate of the data series used to generate the logistic map. Bifurcations are clearly identifiable as precipitous drops in the recurrence rate. Further, one can see that the inverse of the recurrence rate
corresponds to the period of a trajectory in the map (i.e., period two for \(3 < r < 3.449\) has an RR of 0.5, period three at \(r \approx 3.828\) has an RR of 0.33, etc.). This analysis of \(T = 1/RR\) is only valid for non-chaotic portions of the map. The kinks at the period doubling bifurcation points at \(r \approx 3.449\) and \(r \approx 3.544\) are due to transients, thus showing \(\varepsilon\) to be small and a method for determining its size from the data. In an ideal case the transients would not be in the data series and \(\varepsilon\) could be taken to be extremely small, giving a very accurate reading of the period of the orbits of the logistic map.

From the RR in Fig. 4.3, one can easily calculate one of the universal scaling numbers of the map by taking the ratio of the same-period line lengths. Feigenbaum’s constant is given by

\[
\delta = \lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}},
\]

where \(r_n\) is the value of the parameter \(r\) for which there is a \(2^n\) periodic orbit. In Fig. 4.3, the length of any persistent horizontal line is approximately \(r_n - r_{n-1}\). By dividing the length of this line by the next-shortest line of a lower RR, one obtains an estimate of \(\delta\). This example illustrates some of the utility in analyzing data using recurrences.
Figure 4.1: Some common systems (left) and their respective recurrence plots (right). (a) The red curve is a single quasiperiodic orbit of the standard map, (c) logistic map trajectory for \( r = 4 \), and (e) is the Lorenz attractor.
Figure 4.2: Density of recurrences in an RP (recurrence rate) versus ‘time’ at different threshold values $\varepsilon$
Figure 4.3: Recurrence rate superimposed over the data it is calculated from (logistic map), both as a function of the parameter $r$.  

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Chapter 5

Methods

The methods I devised to find the stickiness exponent $\gamma$ derive largely from Refs. [12] and [2]. The former proposes methods based on recurrence plots (namely, recurrence rate), while the latter is based on the traditional definition of recurrence based on Poincaré’s theorem. It can be argued that recurrence plots offer more of a global survey of recurrences as compared to recurrences obtained from comparison of a single point, which is the method used by the authors in Ref. [2]. With recurrence plots there are $N(N - 1)/2$ recurrence comparisons versus $N$ as is traditional in Poincaré recurrence. In the cases where I am comparing data to values obtained by other authors, my data series are of length $4 \times 10^7$, which gives on the order of $10^{14}$ unique comparisons. Initial conditions are randomly chosen in the large stochastic regions.

I use my own code (see appendices) except for calculations of recurrence rate, which I obtain from an already-coded program from Norbert Marwan [17].

5.1 Measuring Stickiness with Recurrence Rate

Since sticky events represent segments of an overall trajectory that has dynamics that are much more regular than those of the overall trajectory, one would expect there to be a pronounced increase in the number of recurrences. In order to quantitatively measure the recurrences at different points in the
trajectory, a windowed recurrence rate will be used. Figs. 4.2 and 5.1 are examples of such an analysis, where the pronounced upswing in values is a clear indication of a sticky event. In Fig. 5.1, the red line is a threshold chosen for defining sticky events. It was chosen as 1-10% above the average level of minimum values sampled over the data series in windows on the order of 1% of the length of the data series. The threshold was very data-dependent. For regions that had a large number of transport barriers, which usually correlated to longer sticky events and a smooth RR outside of sticky regions, \( \varepsilon \) was chosen to around 5% (all the cases in Fig. 5.1). For data that showed a lot of fluctuations (appearing to be ergodic), \( \varepsilon \) had to be chosen to be a larger value.

After choosing a suitable threshold, determining the length of the sticky event, then, is finding the two times that the recurrence rate crosses the threshold for an event: one going in, one going out (see Fig. 5.1). After taking the difference between these two values for all sticky events, one assembles a set of sticky durations,

\[
T = \{t_1, t_2, \ldots \}.
\]  

(5.1)

5.2 Obtaining the Cumulative Distribution Function for Sticky Events

Obtaining the cumulative distribution function involves analyzing the probability of a sticky event of length \( t > \tau \) for \( \tau \gg 1 \) from the following function:

\[
\rho (t > \tau) = \sum_{t=\tau}^{\infty} P(t).
\]  

(5.2)
Multiple authors (notably Refs. [23] and [2]) have found evidence to support a view that cumulative distributions of sticky durations obey asymptotic power laws for phase spaces that are inhomogeneous (see Sec. 3.3). Based on the property of self-similarity an asymptotic power law seems reasonable. These laws take the form

$$\rho(t > \tau) \sim \tau^{-\gamma},$$

(5.3)

while, if the phase space does not have a significant transport barrier (e.g., if cantori are significantly atrophied from $k_g$ levels), then the statistics obey something closer to

$$\rho(t > \tau) \sim \frac{1}{\tau^\beta} e^{-\frac{\tau}{\tau^\beta}}.$$

(5.4)

5.3 Obtaining $\gamma$ from Cumulative Distributions

The two main cumulative distribution functions that are considered are exponential and algebraic. Inferring the statistics from the plots can be accomplished by replotting the data on log plots of different types. The exponential will appear linear on a semi-log plot:

$$\log (\rho_{\text{exp}} (\tau)) = -\frac{\tau}{\tau^\beta} - \log \tau^\beta.$$

(5.5)

An algebraic distribution becomes

$$\log (\rho_{\text{alg}} (\tau)) = -\gamma \log (\tau)$$

(5.6)

and appears linear on a log-log plot.
5.4 Conclusion

The combination of these methods in most cases allows for a straightforward interpretation of the distribution of sticky durations in the data series used for this paper. The statistics at the tails is poor oftentimes and there are some linear fits that do not lend themselves to either a log-log or semi-log plot so other techniques, whether it be longer data series for more accurate fits or a different fitting altogether, may be needed. Results are summed up in the following chapter. The computer code, written in Matlab, appears in the appendices.
Figure 5.1: A sample of the recurrence rate (blue data) for three different chaotic orbits with different values of $k$ for the standard map. The red line represents the threshold for a sticky event. As is evident in the plots, the values of $k$ away from the critical threshold $k_g$ have a very different character.
Chapter 6

Results

6.1 Standard Map

6.1.1 Comparison with Previously Measured Results

The main results of quantification of stickiness come from Refs. [4], [5], [12], and [24]. Below is a chart of values for $\gamma$ from these papers as well as values calculated as part of the research for this thesis (the last column).

Table 6.1: A comparison of stickiness exponents $\gamma$ from different authors. (*) Zaslavsky used $k = 6.908745$ while the last column is calculated using $k = 6.9009$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Chirikov</th>
<th>Zou</th>
<th>Zaslavsky</th>
<th>PAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.971635</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td>1.488</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>1.924</td>
<td>-</td>
<td>1.874</td>
</tr>
<tr>
<td>6.9(*)</td>
<td>-</td>
<td>-</td>
<td>3.974</td>
<td>3.779</td>
</tr>
</tbody>
</table>

As can be seen in the above table, there is general agreement between my numbers and those of Zaslavsky and Zou but my results differ from those of Chirikov. Instead of $\gamma$ becoming smaller from $k_0$ to $k = 1.5$ (i.e., becoming stickier), as Chirkov found, my results show the opposite trend, although the change in values is modest. Other models such as in Ref. [19] use similar methods and find similar results but these are largely parameter-independent. According to my model the parameter of the map plays a crucial role in the
statistics and therefore the overall stickiness. Ref. [2] lists values of \( \gamma \) ranging from 1 to 2.5. In all the literature there is not a consistent method for measuring stickiness, but the methods used by Zaslavsky (based on ‘residence time’) and those introduced by Zou seem consistent and my results give close approximations of their values.

Of all the plots in Fig. 6.1, the cumulative distribution for sticky durations with \( k = k_g \) seems to lend itself least to linear fits on full logarithmic or semi-logarithmic plots. The two data series generated, for randomly chosen initial values, have similar behavior which does not seem to lend itself to either of the plots. A notable feature that complicates the shape is the kink around \( \log \tau \approx 5 \) which suggests two different scaling regions since the graph before this value looks similar to the graph after it. These scaling regions are apparent in other distributions. Notable examples are for \( k = 1.5 \) and \( 1.8 \) which have two scaling regions in the full logarithmic plots. The slopes for larger \( \tau \) match roughly what other authors have found but the regions of \( \log \tau \) from 0 to 2.5 have slopes on the order of \( 10^{-2} \) to \( 10^{-1} \), indicative of extremely sticky behavior for times \( \tau \sim 1 - 12 \) (or \( t \sim 2500 - 30000 \) in the time units of the standard map).

To check the consistency of the results, the time set [Eq. (5.1)] was shuffled and chopped into sets of different lengths spanning from 10% of the total length of the original time set to the full time set. The results of this analysis yielded values of \( \gamma \pm .4 \), where \( \gamma \) is the value calculated from the full time set.
6.1.2 Stickiness as a Function of $k$

A goal of this research was to find the $k$ dependence of $\gamma$. Chirikov gives higher values (less sticky) of $\gamma$ near criticality and lower values (stickier) when $k$ leads to global chaos. In general there has been little speculation about the behavior of $\gamma$ outside these two considerations. It was previously speculated in this paper that there are multiple statistical regions of the standard map. A partial answer is given below (see Fig. 6.2(a)). Notable values are those at $k = 1.2$ and $k \approx 8\frac{8}{9}$ which displayed behavior consistent with ergodic orbits. All other $k$ values sampled seemed to lend themselves to an algebraic distribution.

6.2 Standard Nontwist Map

Early analyses seemed to show little, if any, recurrent behavior consistent with stickiness for the SNM with $(a, b) = (0.686049..., 0.742494...)$, which contains the golden critical torus [see Fig. 2.2(b)]. Another map, Fig. 6.3, with more significant transport boundaries (including island chains and an invariant curve with winding number of the inverse silver mean $[0; 2, 2, ...]$), show stickiness of $\gamma \approx 2.4$ [Fig. 6.3(c)]. A possible explanation for the relatively high $\gamma$ (low sticky) value is the unboundedness of the momentum component in the map. Fig. 6.3(d) shows a probability distribution of the $y$-coordinate of a segment of the orbit in Fig. 6.3(a) [the $x$ component has a probability distribution (not shown) that is level across the $x$ domain]. As can be seen, there are preferred regions for momentum, oftentimes far away from the transport boundaries near the origin. In the SM, there are stochastic layers because of large transport boundaries, thus increasing the likelihood of a chaotic segment to become stuck to a stable edge. This is not the case with the SNM.
Figure 6.1: (a), (c), (d) Logarithmic plots of the cumulative distribution functions of the duration of sticky events for different values of $k$ for the standard map. (b) Semi-log plot of the cumulative distribution of sticky events for $k = 1.2$. In all the plots, blue lines are from trajectories of length $2 \times 10^7$, while red lines are for $4 \times 10^7$ data points. One unit of $\tau$ represents a window size of 2500 units of time from the standard map trajectory.
Figure 6.2: (a) A plot of $\gamma$ versus $k$ for all the values calculated for this thesis (blue line), as well as some of the values from other authors listed in Table 6.1.1 (black points with red border). (b) An example of how fits are obtained for the cumulative distribution data. Fits are displaced slightly from data for sake of clarity.
Figure 6.3: (a) A section of SNM where the blue triangle is the initial condition and the red is the trajectory analyzed for stickiness in (b). (c) Cumulative distribution of the RR data which gives a linear slope of $\approx 2$. (d) Probability distribution of the $y$ coordinate in the SNM.
Chapter 7

Further Research

Below are directions I wish to take my work.

• An analysis of longer data series for the maps explored above so that more accurate values of $\gamma$ may be provided.

• A detailed plot showing the dependence of $\gamma$ on the kicking parameter $k$ for the standard map.

• An extension of these methods to the 2D parameter space of the standard nontwist map.

• An investigation into the correlations, if any, there are between the measure $A_{IT}$ and the regions of stickiness (see Sec. 3.2), and an extension of this analysis to nontwist maps.

• An extension of my methods to try to replicate the results that others have received for maps such as the separatrix map or with different distribution functions.

• An analysis of other measures of stickiness such as the total time on a sticky event divided by the overall time of a trajectory.
Appendices
Appendix A

Standard Map Code

Following is the code written to produce the orbits for the standard map analyzed in this thesis as well as surfaces of section as seen in Fig. 2.1.

```matlab
%matlab script that creates surfaces of section for standard map as well as trajectories used in this thesis for recurrence rate analysis
k = 1.2; % .971635406;
twopi = 2*pi;
iter = 2E7; %iterations of orbit to be analyzed with RR
m = 20; %number of orbits to fill surface of section
iterx = 1E3; %iterations of the 20 above orbits
yes = 1; % decides whether or not to output data
x = zeros(iterx+1,m); %sets aside space in memory for iterations
y = zeros(iterx+1,m);
q = zeros(iter,1);
p = zeros(iter,1);

%generates the map by initiating m orbits with random initial conditions
if yes
    for j=1:m
        x(1,j)= rand(1);
y(1,j) = rand(1);
        for n=1:iterx
            y(n+1,j) = mod(y(n,j) + k/(twopi)*sin(twopi*x(n,j)),1);
x(n+1,j) = mod(y(n+1,j) + x(n,j),1);
        end
    end
end

%q and p are the orbits that are analyzed for this thesis
q(1) = .15; %initial values
p(1) = .08;

for n=1:iter
    p(n+1) = mod(p(n) + k/(twopi)*sin(twopi*q(n)),1);
    q(n+1) = mod(q(n) + p(n+1),1);
end
```
if yes
figure, hold on;
plot(x,y,'k.','markersize',1);
plot(q(1:2E6),p(1:2E6),'r.','markersize',1);
plot(q(1),p(1),'g''','markersize',15);
axis square, title(sprintf('k = %f',k));
end

if yes-1 % exports data if yes = 0
    data = [p q];
    save data/data2E7_k1p0.dat data -ascii -tabs
end

clear iter* m k twopi j n symline z x y yes q p
Appendix B

Sticky Duration Code

Following is the Matlab code that I wrote, followed by its dependent function, to process the recurrence rate data.

function rho = rhotimes(rrdata)
    timeset = rectimes(rrdata,10,1);
    times = nonzeros(timeset);
    maxt = max(timeset);
    maxl = length(timeset);
    rho = zeros(maxt,1);
    for ii = 1:maxt
        for jj = 1:maxl
            if times(jj)>=ii
                rho(ii) = rho(ii) + 1;
            end
        end
    end
    rho = rho/maxl;
    t = 1:length(rho);
    logrho = log(rho);
    logrho = log(nonzeros(rho));
    logrho = logrho;
    logrhotime = log(t);
    plot(logrhotime,logrho,'b-','markeredgecolor','r','markerfacecolor','r');

Begin function rectimes.

%times = a given RR time series
% times = input windowed recurrence rate data
% bindiv = size of set of sticky durations/binsize
% type = 1 for duration of sticky events; = 0 for duration between sticky
% events.  defaults at 1.

function timeset = rectimes(times,binnum,type)
if nargin < 3; type = 1; end %defaults to measure durations
if nargin < 2; binnum = 10; end %defaults to 10 bins

k = 1E3;
rrmins = zeros(fix(length(times)/k),1);
for ii=1:fix(length(times)/k)
    rrmins(ii) = min(times(((ii-1)*k+1):(ii*k)));
end
thr = 1.05*mean(rrmins); %threshold value set at 5% above average
enterr = zeros(length(times),1); %sets aside memory for variables
exitt = zeros(length(times),1);

j=1;
m=1;
for i=2:length(times)
    %finds when the threshold enters sticky region
    if times(i)>thr && times(i-1)<thr
        enterr(m) = i;
        m=m+1;
    end
    %finds when threshold leaves sticky region
    if times(i)<thr && times(i-1)>thr
        exitt(j) = i;
        j=j+1;
    end
end
enterr = nonzeros(enterr);
exitt = nonzeros(exitt);
if exitt(1) < enterr(1)
exitt = exitt(2:end);
end
if enterr(end) > exitt(end)
    enterr = enterr(1:(end-1));
end

'data' contains the set of duration times for sticky events
if type
    data = exitt - enterr;
elseif type == 0
    data = enterr(2:end) - exitt(1:end-1);
else
    data = exitt - enterr;
end
timeset = data;
%% End of function
Bibliography


Vita

Peter Andrew Eschbacher was born August 22, 1982 and grew up in Kansas City, Missouri. He graduated from Park Hill High School in 2001 and went on as a physics major at the University of Missouri-Columbia. After three years there, and intending to come back, he studied at the University of Edinburgh in Edinburgh, Scotland. Soon after he transferred and was accepted into the MPhys program. In Spring 2006 he received the MPhys degree from the University of Edinburgh. Upon graduating, he attended the University of Texas at Austin in 2006 as a graduate student and is working towards a Master of Arts degree. In December 2009 he plans on finishing the UTeach program and receiving a teaching certificate for physics education.

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