Chaos and Graphics

A note on on–off intermittency in a chaotic coin flip simulation

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Abstract

Presented is a computer simulation of a variation of the Gambler’s Ruin game that is used to model intermittent chaos. A rich player gambles with a set amount of money \( m \). The poor player starts out with zero capital, and is allowed to flip a coin in order to try to win the money. If the coin is heads, the poor player wins a dollar but if it is tails, the poor player loses a dollar. The length of time it takes for either player A or player B to reach zero is recorded for a specified number of games. The simulation presents a model for intermittent chaos, as it manifests quiescent or periodic behavior alternating with chaotic bursts.

\[ X_{n+1} = X_n + g_n \]  

1. Introduction

The gambler’s ruin game involves a random walk problem with numerous applications in such various fields as genetics and quantum mechanics. The classical gambler’s ruin game proved to have fractal characteristics. Coin-toss problems in which money is exchanged are studied by economists, due to the possible theoretical models they provide.

Intermittency is characterized by periodic or almost-periodic behavior interspersed with chaotic behavior that can be in long or short bursts. On–off intermittency is characterized by extended periods of stasis that are followed by bursting [1]. Quiescent behavior followed by sharp outbursts is very common in nature, and volcano eruptions and earthquakes are a few examples of such behavior. Chaotic bursting as a research topic is of interest in areas such as astronomy [2], financial markets [3], lasers [4], neurobiology [5], and circuits [6,7].

The present research simulates on–off intermittency via a variation on the gambler’s ruin random walk problem, which I call the “Y variation” [8] after its creator, James Yorke. In the original Y variation, the length of the game is not determined by the amount of money the gambler has lost, but is predetermined by the amount of flips allowed. The Y variation has two players, with player A representing a rich man with a set amount of money \( m \) that he wishes to gamble, and player B, a poor man. The poor man is allowed to try to win this money by flipping an unweighted coin \( n \) times so that the game ends only when \( n \) does. (In other words, that even when the poor man loses all of his money, he is allowed to keep flipping until \( n \) is reached.) The poor man wins a dollar if the coin is heads, and loses a dollar if the coin is tails.

The generating equation that may be used to represent the game is

\[ X_{n+1} = X_n + g_n \]  

with \( g_n = 1, -1, \) or 0. This equation thus represents the velocity, or incremental movement of the money. This motion is different from the classical gambler’s ruin, where \( g_n \) has values equal to 1 or \(-1\), only. Using standard time-series analysis tests of power spectra, Poincare sections, and Lyapunov exponents, the sequences generated are found to be chaotic.

The current paper presents a modification to the Y variation. Instead of imposing a limit on the maximum number of flips player poor man is allowed, here a limit is imposed on the total number of games the poor man is allowed to play. Thus, the current paper examines the length of a series of games played. This model results in nearly flat behavior followed by bursting, which yields fractals in the shape of triangular bifurcations. Viana et al. [9] investigated on–off intermittency models using a biased
random walk model to study shadowability breakdown in systems with chaotic bursting has been studied by. Shadowability breakdown occurs when a chaotic trajectory is no longer close to and is no longer deformable to another chaotic trajectory.

2. On–off intermittency and the Y variation

Classical Brownian motion has a mean of zero and a unit variance. These conditions do not hold for the Y variation, where the both the mean and the variance are positive and increasing. Chaos arises in the Y variation as a result of fluctuations in the variance [8]. Fig. 1 is a phase space plot of the variance for a game with a sequence of 1 million points, where the money gambled is limited to $1000. The lower states are more densely populated than the higher ones, with the latter rarely being visited in comparison. Furthermore, the figure shows regions where the trajectories bifurcate, undergoing doubling, tripling, and even quadrupling before eventually settling onto the main attractor. Recurrence plots are diagnostic tools that display time correlation behavior that is otherwise inaccessible [10]. A recurrence plot of the variance also suggests chaotic behavior (Fig. 2). The horizontal and vertical lines displayed show times where the states are invariant or very slow to change, which is a characteristic of intermittency [11]. Recurrence plots for financial time series such as residuals in macroeconomics [12] and the Dow Jones industrial index and the NIKKEI 500 typically show similar behavior [13].

For the present research, a modification of the original Y variation is introduced. A computer program simulates, as before, two players gambling with a set amount of money $\text{m}$. Player B, the poor man, starts off with a dollar. This time however, a game ends whenever player A or B loses all of the money. The number of flips taken before player A or B loses is recorded, and this gives the length of time each game is played. The total number of games is predetermined by an amount $\text{n}$, which takes on the values $10^2$, $10^4$, $10^5$, or $10^6$. An example for a game with $10^4$ sets is represented in the plot in Fig. 3. The $y$-axis represents the length of games or number of flips, and the $x$-axis represents the total number of games played. Most of the
games end with one toss of the coin. Occasionally the games last beyond a few flips, with the longest one recorded at 10^6. The simulation has regular behavior interspersed with bursting, and hence simulates on–off intermittency. The inset graph in Fig. 3 provides a view of one of the bursts. There is a marked similarity to the tent map or chaotic sawtooth wave insofar as the triangular appearance is concerned. However the height, frequency, width, and location of the bursts vary within the graph. Outside of the bursts, the periodic part is a main attractor, which is an invariant hyperplane, and has a value equal to one. The other games show the same type of behavior, though the bursting parameters are different, thus displaying sensitivity to the initial conditions.

Fig. 4 is a phase space map for the trajectory of Fig. 3. This strange attractor represents a series of triangles that bifurcate by doubling, tripling, quadrupling and then splitting several more times, before settling onto the main attractor. The phase map shows more than one attractor, with orbits visiting certain states more than once, as evinced by the thickness of the lines. The map displays a shift amongst both axes that is properly displayed in an

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![Fig. 3](image1.png)

Fig. 3. Plot of a typical coin flip simulation in the current research.

![Fig. 4](image2.png)

Fig. 4. Phase space map for the trajectory in Fig. 3.
embedding dimension of three or more. This attractor has the same shape for all of the games, though as before, the parameters are different.

Fig. 5 is a Poincare section of Fig. 4. The attractors are in the form of an L shape with the y-axis separated into distinct groups. The L shape becomes denser on both axes, and the attractor grouping more pronounced as \( n \) increases (not shown).

3. Applications

The amount of money in this simulation is arbitrarily set to $1000, but, aside from changing this limit, rescaling can be introduced to select the region of interest. This, coupled with filtering to suppress noise and/or unwanted frequencies, can be used to retain the basic attractor characteristics while altering the phase space dynamics. If one rescales the
sequence Fig. 3 by multiplying by $10^{-6}$, and then subjecting it to a low-pass filter, the adjusted signal in Fig. 6 and the phase space in Fig. 7 results. The set is limited to values between 0 and 1, and the chaotic bursts are in the shape of gamma distributions. As before, the phase space has several attractor regions, but the angle of the L-shaped attractor is more acute, so that the bottom ends at a point, and the attractor shapes are nearly elliptical.

4. Conclusion

This work presented a novel approach for modeling on-off intermittency by using a variation on the gambler’s ruin problem. In this work, the Y variation uses $n$ to represent a collection of games, where the value that is recorded for analysis is the length of the individual game. The signals that are output from this modification simulate on-off intermittency, where relatively long periods of quiescent or periodic behavior are suddenly interrupted by chaotic bursts. These signals, instead of being stochastic, are chaotic, and are in the form of strange attractors.

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References