Synchronous patterns and intermittency in a network induced by the rewiring of connections and coupling

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ABSTRACT
The connection architecture plays an important role in the synchronization of networks, where the presence of local and nonlocal connection structures are found in many systems, such as the neural ones. Here, we consider a network composed of chaotic bursting oscillators coupled through a Watts-Strogatz-small-world topology. The influence of coupling strength and rewiring of connections is studied when the network topology is varied from regular to small-world to random. In this scenario, we show two distinct nonstationary transitions to phase synchronization: one induced by the increase in coupling strength and another resulting from the change from local connections to nonlocal ones. Besides this, there are regions in the parameter space where the network depicts a coexistence of different bursting frequencies where non-stationary zig-zag fronts are observed. Regarding the analyses, we consider two distinct methodological approaches: one based on the phase association to the bursting activity where the Kuramoto order parameter is used and another based on recurrence quantification analysis where just a time series of the network mean field is required.

Synchronization of networks has been numerically explored since the last century where connection architecture plays an important role. Here, we study the effects of the rewiring connection probability and coupling strength in a system composed of bursting oscillators simulated through the Rulkov model under local and nonlocal couplings using Watts-Strogatz networks. The Kuramoto order parameter, computed over the individual bursting behavior, and the recurrence quantification analysis based on the network mean field are used to investigate the phase synchronization and intermittency characteristics of the network. We divide the parameter space into four regions regarding its dynamical properties and synchronization characteristics. Despite the well-known phenomena of phase synchronization occurring for large values of the rewiring connection probability and the coupling parameter, there are open questions about the details of the dynamical behavior for intermediate values. In this scenario, we have observed the existence of zig-zag fronts and metastable states with intermittent behavior as a result of the interplay between coupling and connection architecture.

1. INTRODUCTION
Nonlinear dynamical features of complex systems are useful to investigate many problems of biology, physics, social science, and engineering. Particularly, mathematical and computational approaches have been used since the last century in order to clarify the behavior of these systems. Particular interest has been given to coupled oscillator models, described by differential equations or discrete maps, which can be used to simulate different phenomena such as neural activity and neural behavior.

In the scenario of coupled oscillators and complex networks, synchronization phenomena are very important and have been observed in many fields of science. In neural networks, the synchronization phenomena are associated either with regular brain functioning or neurological disorders related to the lack or excess of the synchronization level. In general, emergent behaviors, such as nonstationary transitions to synchronization, including two-states-on-off intermittency, are important features of complex systems, where the global behavior is different, usually richer than the sum of individual ones. Besides, the
connection architecture may play an important role in the route to synchronization.

Several works have investigated the synchronization and intermittent behavior of networks under different circumstances. In some cases, two topologies may be considered: the Watts-Strogatz network, where the connection scheme is characterized by a local structure evolving to a small-world network and finally a random one as the rewiring probability increases; and the Newman-Watts network, where initially, the network has a local connection structure and by the addition of nonlocal connections, it turns into a small-world one. In general, it is known that for regular networks, where most of the connections are local, the phase synchronization is not observed. However, diagonal structures and zig-zag fronts are observed in the space-time plot to these scenarios, considering different local dynamics. Additionally, increases in the coupling strength or the number of nonlocal connections make the network to transit to phase synchronization.

Here, in order to investigate the topology influence over the synchronization characteristics, we have considered a network composed of chaotic-identical-bursting oscillators in the Watts-Strogatz networks since small-world and random topology characteristics are observed in real systems such as the neural ones. The bursting behavior is characterized by a sequence of spikes followed by a resting time. In these regimes, two time scales are important: a slow one related to the bursting frequency and a fast one related to the spike activity. The individual behavior is simulated through the Rulkov map. The first approach to investigate the synchronization and intermittent behavior consists of the phase association to the individual bursting activity and the use of the Kuramoto order parameter. As a second approach, we have used the recurrence analysis, which is based on the network mean field.

The synchronization and intermittency characteristics are studied in the parameter space defined by the probability of rewiring a connection of the network (p) against the coupling strength (ε). In this way, the connection architecture is initially composed of a second neighborhood structure, and the increase of p results in the rewiring of a local connection into a randomly chosen one. Here, we have observed different regions in this parameter space regarding synchronization characteristics: nonsynchronized behavior is observed for smaller values of the coupling strength; zig-zag fronts are observed for higher values of coupling strength but small values of p, where intermittent behavior is noticed; and phase-synchronized states are observed for higher values of ε and p. Besides this, we have noticed two distinct transitions to phase synchronization. The increase of coupling strength changes the network dynamics from nonsynchronized to phase-synchronized states, and the increase of p makes the network transits from zig-zag fronts to phase-synchronized states. For both cases, intermittent behavior is observed.

The paper is organized as follows: in Sec. II, details of the Rulkov map and its dynamical behavior are presented and bursting is defined; in Sec. III, the coupling architecture and the Watts-Strogatz networks are described; in Sec. IV, the methodology and the synchronization quantifiers are presented; the results and discussions are given in Sec. V followed by the conclusions given in Sec. VI.

II. THE MODEL

The Rulkov model is used to simulate the chaotic bursting behavior, which can be described by the two-dimensional map:

\[
x_{i+1} = \frac{\alpha}{1 + x_i^2} + y_i + I_{i,t},
\]

\[
y_{i+1} = y_i - \beta x_i - \gamma,
\]

where \(x_i\) is the fast variable and \(y_i\) is the slow variable of the \(i\)th oscillator (neuron). The parameters \(\alpha = 4.1, \beta = \gamma = 0.001\) are chosen in order to obtain the bursting behavior. The last term in the first equation characterizes the coupling term \((I_i)\), which can be understood as the influence of the other neurons,

\[
I_{i,t} = \frac{\varepsilon}{N} \sum_{j=1}^{N} a_{ij} x_{j,t},
\]

where \(\varepsilon\) is the coupling strength, \(\chi\) is a normalization factor (average number of connections in the network), and \(N\) is the number of neurons in the network. \(a_{ij}\) is the adjacency element of the connection matrix, where \(a_{ij}\) assumes 1 (0) if \(i\) and \(j\) are (not) connected.

The dynamical behavior produced by the model is depicted in Fig. 1, where the black line represents the bursting (fast variable) and the magenta line shows the slow variable as a function of the discrete time \(t\). The maxima of \(y\) indicate the beginning of the bursts and used to build a phase for each oscillator. The initial conditions are randomly chosen in the interval described by each variable as depicted in Fig. 1.

III. THE CONNECTION ARCHITECTURE

The connection architecture plays an important role in the dynamical properties of complex networks. Regarding neural systems, the topology can be more important than the physiologic characteristics of neurons. Here, we consider the connection scheme based on Watts-Strogatz networks, starting from second neighborhood regular structures, where there are just local connections, and rewiring some of these connections by random ones following a probability \(p\). The increase in \(p\) makes the network to transit from a regular to a small-world network; however, if \(p\) is further increased, a random network is obtained. A visual example of this phenomenon is mimicked in Fig. 2, where just 15 nodes are considered. Panel
(a) represents the regular second neighborhood network, panel (b) a small-world network, and panel (c) a random one.

It is possible to evaluate the characteristics of the coupling scheme as a function of the rewiring connection probability \( p \) in order to obtain more details of the network’s topology. As observed by Watts and Strogatz, the small-world networks depict regular and random characteristics. These networks present a high value of clustering coefficients and a small value of average path length, as defined in Ref. 2. On the other hand, Latora and Marchiori have defined the efficiency of the networks in order to study the small-world ones. The efficiency in the communication between nodes \( i \) and \( j \) is inversely proportional to the shortest distance between \( i \) and \( j \) \( (d_{ij}) \).

For this way, the global efficiency of the network is defined as \( E_{\text{global}} = 1/N(N-1) \sum_{i \neq j} 1/d_{ij} \). Furthermore, the local efficiency of the network is defined as the average efficiency of the local subgraph \( E_{\text{local}} = 1/N \sum_{i \in G} E(G_i) \), and it is related to the efficiency of the communication between the first neighbors.

Figure 3 depicts the global and local efficiency of the network (magenta and black lines, respectively). For the interval \( 1 \times 10^{-2} < p < 3 \times 10^{-3} \), the global efficiency increases and the local efficiency presents a high value, which characterizes a small-world network. For \( p < 1 \times 10^{-3} \), the network is essentially regular where just \( E_{\text{local}} \) depicts a high value, and for \( p > 3 \times 10^{-1} \), the network presents random topology characteristics, which follows the original idea of Watts and Strogatz.

**IV. METHODOLOGY**

To analyze the phase synchronization, in this case, is necessary to associate a phase to each oscillator. As observed in Fig. 1, each maximum of \( y \) indicates the burst beginning, so the phase is increased by \( 2\pi \) every time \( y \) reaches a maximum, and a phase continuous variation can be obtained using

\[
\theta_i(t) = 2\pi k + 2\pi \frac{t - t_{kj}}{t_{kj} - t_{ki}}, \quad t_{ki} < t < t_{kj+1},
\]

where \( t_{kj} \) is the time when the \( k \)th maximum of \( y \) occurs, which means the beginning of the \( k \)th burst of the \( j \)th oscillator. The individual burst behavior is maintained for all values of coupling strength \( \varepsilon \) and rewiring connection probability \( p \). In this way, the phase can be obtained from the \( y \) variable without reconstruction of the dynamics in the higher dimensional phase space.

Since the phase is associated with the bursting activity, the bursting frequency can be described as

\[
\omega_i = \lim_{t \to \infty} \frac{\theta_i(t) - \theta_i(0)}{t}.
\]

Besides this, it is possible to analyze the phase synchronization of the network through the Kuramoto order parameter,

\[
R(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right|,
\]

where \( \theta_j \) is the phase of the \( j \)th oscillator and \( N \) is the number of oscillators in the network. Here, a network in a phase-synchronized state depicts \( R \to 1 \), while a not synchronized case depicts \( R \to 0 \).

The mean value of the Kuramoto order parameter is considered to evaluate the synchronization level as a function of coupling strength \( \varepsilon \) and rewiring connection probability \( p \),

\[
\langle R \rangle = \frac{1}{t_f - t_0} \sum_{t=t_0}^{t_f} R(t),
\]

where \( t_f \) and \( t_0 \) are the simulation and transient time.

On the other hand, it is possible to use the recurrence quantification analysis to investigate the synchronization characteristics using a time series of the dynamical system, without a phase association.

The original approach performed by Eckmann et al. was based on visual forms to investigate dynamical systems. However, recent works have shown the existence of mathematical structures in the recurrence plots. In this way, the recurrence matrix can be defined as

\[
R_{ab}(\delta) = \Theta(\delta - ||x_a - x_b||), \quad a, b = 1, 2, \ldots, T,
\]

where \( \Theta \) is the Heaviside function, \( \delta \) is the recurrence threshold, and \( T \) is the size of the time series analyzed. Here, if the state \( a \) is recurrent to the state \( b \), \( R_{ab} = 1 \) (a black dot in the recurrence plot). On the other hand, if \( a \) is not in \( \delta \) vicinity of \( b \), \( R_{ab} = 0 \) (a white dot in the recurrence plot).
Here, we use the time series of the network mean field, defined as
\[
\bar{x}_i = \frac{1}{N} \sum_{t=1}^{N} x_i(t),
\]
where \( x \) is the fast variable of the Rulkov map described by Eq. (1).

The mean field of a phase-synchronized state depicts an oscillatory behavior. Therefore, it is possible to consider the determinism \((\Delta)\) of the time series, which considers the diagonal lines in the recurrence matrix, to study the synchronization characteristics. A diagonal line of length \( \ell \) is understood as a segment of the trajectory rather close to another trajectory segment during \( \ell \) time steps in a different time.

The determinism is defined as
\[
\Delta(\ell_{\text{min}}, \delta) = \frac{\sum_{\ell = \ell_{\text{min}}}^{\ell_{\text{max}}} \ell P(\ell, \delta)}{\sum_{\ell = 1}^{\ell_{\text{max}}} \ell P(\ell, \delta)},
\]
where \( \ell_{\text{min}} \) is the minimum diagonal line length considered and \( P(\ell, \delta) \) is the probability distribution function (PDF) of the diagonal lines.

The mean value of the determinism is considered to analyze the network features as a function of \( \varepsilon \) and \( p \),
\[
\langle \Delta \rangle = \frac{1}{t_f - t_i} \sum_{t=t_i}^{t_f} \Delta(t).
\]

V. RESULTS AND DISCUSSIONS

We consider a network composed of \( N = 1000 \) chaotic-identical-bursting oscillators in the space parameter defined by \( p \times \varepsilon \), where \( p \in [1.0 \times 10^{-3}, 1.0] \) and \( \varepsilon \in [0.0, 0.1] \). In this scenario, each neuron has, on average, 4 connections, which leads to a network with 4000 connections with a \( \chi = 4 \) normalization factor. Smaller values of \( p \) are not considered since there is no rewiring of connections to these cases, and the network for \( p = 0 \) is essentially the same as \( p = 1 \times 10^{-3} \). Bigger values of \( \varepsilon \) are not considered since higher values of external input under each oscillator may change the individual dynamics of bursting.

A. Synchronization characteristics

The main scenario is depicted in Fig. 4, where the synchronization characteristics are analyzed as a function of \( p \) and \( \varepsilon \) by using the Kuramoto order parameter \((R)\) and the determinism of the network mean field \((\Delta)\). The parameter space of \( p \times \varepsilon \) is divided into four regions regarding the synchronization quantifier values. The transient time is given by \( t_0 = 100,000 \), and the total simulation time is \( t_f = 500,000 \). The results depicted are an average of 20 different networks (different connection matrices) for each \( p \) value. The recurrence threshold is set as \( \delta = 0.11 \), which results from the maximization of \( d \Delta(\delta)/d\delta \) since this condition is related to the maximum sensitivity of the determinism. Besides this, the minimum diagonal length is set as \( \ell_{\text{min}} = 40 \) in order to better distinguish the phase-synchronized states from nonsynchronized ones. The determinism is evaluated using overlapped moving windows of 5000 points.

Figure 4(a) depicts the mean value of the Kuramoto order parameter \((R)\). Distinct dynamical behaviors are observed: the blue tones indicate the absence of phase synchronization, the green and yellow tones indicate transition states, and the orange and red tones indicate the phase-synchronized states. For small values of \( \varepsilon \), the network shows a nonsynchronized state for the entire interval of \( p \), which is represented by region (1). Region (2) depicts blue tones, which indicates the absence of phase synchronization. An interesting point here consists of the transition region, denoted as region (3). Two distinct transitions are observed: for higher values of probability \((p > 4 \times 10^{-2})\), the increase of the coupling strength changes the
network dynamics to phase-synchronized states (represented by the horizontal arrow), starting with lower values of the probability \( p < 4 \times 10^{-2} \) and for low values of coupling strength \( \varepsilon > 0.025 \), the increase of \( p \) makes the network to present phase-synchronized states [region (4)] (indicated by the vertical arrow). It characterizes a distinct phenomenon since for a fixed value of \( \varepsilon \), the rewiring of connections is able to change completely the dynamical behavior of the network. A similar phenomenon is observed in other coupled networks.\(^{21,22}\)

Figure 4(b) depicts the mean value of the determinism of the network mean field, which is able to evaluate synchronization.\(^{23,24}\) Here, regions (1) and (4) depict the same information those depicted by the Kuramoto order parameter. However, in region (2), the determinism and the Kuramoto order parameter show distinct behaviors with a high value for \( \langle \Delta \rangle \), while \( \langle R \rangle \) stays small. This fact is related to the characteristic of the determinism. It evaluates the phase synchronization of networks; however, it can also depict a high value where the network presents diagonal structures in the space-time profile.\(^{25,26,27}\) Therefore, in this case, for lower values of the probability \( p < 4 \times 10^{0} \) and higher values of coupling strength \( \varepsilon > 0.025 \), the network shows diagonal structures or zig-zag fronts,\(^{28,29}\) which can be better observed in the raster plot seen in Fig. 5. The smaller values of \( \langle R \rangle \) indicate the absence of horizontal structures or phase synchronization. A similar dynamical scenario was observed in different dynamical systems.\(^{22,23,24,25}\) In this way, the characterization of the parameter space \( (p \times \varepsilon) \) into four regions regarding synchronization phenomena is based on values of \( \langle R \rangle \) and \( \langle \Delta \rangle \).

The raster plot of the fast variable of the network \( \chi_{t} \), obtained using the criteria of \( \chi_{0} = 0.0 \) and the positive first derivative, is used to exemplify the dynamical behavior of each region shown in Fig. 4, including its transitions as indicated by the arrows. Figure 5 depicts representative raster plots for the four regions. The first row [panels (a)–(c)] represents the influence of the increase of \( p \) for a fixed coupling strength of \( \varepsilon = 0.040 \). Panel (a) is representative of region (2), where \( p = 8 \times 10^{-3} \), and the diagonal structures or zig-zag fronts are observed.\(^{25,26}\) Panel (b) exemplifies region (3), where \( p = 2 \times 10^{-2} \), and the transition occurs as a consequence of the connection rewiring. It is observed that diagonal and horizontal structures are the cause of the increase of the determinism value, which is shown in Fig. 4(b). Finally, panel (c) represents a phase-synchronized state of region (4), where \( p = 9 \times 10^{-1} \), and horizontal structures are observed, which indicate the bursting synchronization.

Figures 5(e)–5(g) depict the influence of the increase in coupling strength for a fixed value of probability of \( p = 6 \times 10^{-1} \) [the horizontal line in Fig. 4(a)]. Panel (e) is representative of region (1), where \( \varepsilon = 0.004 \), and a nonsynchronized state is observed. Panel (f) shows an example of region (3) for \( \varepsilon = 0.009 \). Here, the transition to phase synchronization is induced by the increase in coupling strength, where the onset of horizontal structures is observed without the existence of zig-zag structures as observed in panel (b). Finally, panel (g) is representative of region (4) for \( \varepsilon = 0.045 \), and a phase-synchronized state is observed.

To conclude, panels (d) and (h) in Fig. 5 depict the two possible transitions indicated by the arrows in Fig. 4(a). The first one [panel (d)] is related to the increase in \( p \), where a transition from region (2) to region (4) is observed. The second transition [panel (h)] is related to the \( \varepsilon \) influence, where the transition from region (1) to region (4) is observed. The main distinction between them consists of the contrast in the synchronization quantifiers \( \langle R \rangle \) and \( \langle \Delta \rangle \) for \( p < 3 \times 10^{-3} \) [panel (d)] since in this region, zig-zag fronts are observed and \( \langle R \rangle \) is not able to characterize this state. For the case of panel (h), \( \langle \Delta \rangle \) and \( \langle R \rangle \) depict similar behaviors for all \( \varepsilon \).
interval characterizing a transition from a nonsynchronized to a phase-synchronized state.\textsuperscript{11}

A useful quantifier to analyze synchronization in a network of oscillators consists of the bursting frequency, described by Eq. (5). Figure 6 depicts the probability distribution function (PDF) of the neuron’s bursting frequencies ($\omega_i$). Here, $\omega_i$ is evaluated for all neurons in the network in distinct cases: a nonsynchronized state ($p = 1 \times 10^{-2}$ and $\varepsilon = 0.005$), represented by the cyan bars, in which a normal distribution is observed; a case with zig-zag structures ($p = 1 \times 10^{-2}$ and $\varepsilon = 0.080$), represented by the magenta bars, in which a distribution with several peaks is observed, indicating the existence of different groups of neurons with different frequencies ($\omega_i$). Here, a normal test\textsuperscript{11} results in the statistical value of 0.2492 for the nonsynchronized case and 45.0558 for the zig-zag cases. Finally, the phase-synchronized case depicts a great peak indicating that most of parts of neurons show a similar frequency.

![Figure 6](image)

**FIG. 6.** PDF of the neuron’s bursting frequencies ($\omega_i$) for a representative case of a nonsynchronized state (cyan bars) where $p = 1 \times 10^{-2}$ and $\varepsilon = 0.005$, a representative case of zig-zag structures (magenta bars) where $p = 1 \times 10^{-2}$ and $\varepsilon = 0.080$, and a representative case of phase-synchronized states (black bars) where $p = 7 \times 10^{-1}$ and $\varepsilon = 0.050$. The nonsynchronized case depicts a normal distribution, while the zig-zag structures show a PDF with several peaks, indicating the existence of different groups of neurons with different frequencies ($\omega_i$). Here, a normal test results in the statistical value of 0.2492 for the nonsynchronized case and 45.0558 for the zig-zag cases. Finally, the phase-synchronized case depicts a great peak indicating that most of parts of neurons show a similar frequency.

A test to compare the distribution with a normal one is applied to the cases of nonsynchronized state and zig-zag structures. The test is based on the ideas of D’Agostino and Pearson,\textsuperscript{11} and it is performed using the SciPy library.\textsuperscript{11} In this sense, the test indicates a statistic value of 0.2492 for the nonsynchronized case, in which normal distribution is observed (cyan), and 45.0558 for the zig-zag cases (magenta), which leads to the conclusion of the differences between the two cases.

This scenario can be investigated in the parameter space of $p \times \varepsilon$. As observed in Fig. 5, the zig-zag structures depict formation in groups. In order to quantify this phenomenon, the network can be divided into different groups, and the average bursting frequency for each group is evaluated by\textsuperscript{30-31}

$$\Omega_j = \frac{1}{N_j} \sum_{i \in j} \omega_i, \quad (12)$$

where $j$ indicates the group of oscillators, $N_j$ is the number of oscillators in each group, and $\omega_i$ is the bursting frequency of the $i$th oscillator. Therefore, the standard deviation over the values of mean frequencies ($\Omega_j$) brings information about how distinct the groups’ mean frequencies values are,

$$\bar{\Omega} = \sqrt{\frac{1}{N_{\text{group}}} \sum_{j=1}^{N_{\text{group}}} (\Omega_j - \bar{\Omega})^2}. \quad (13)$$

Here, $\bar{\Omega}$ is the average over the $N_{\text{group}}$ groups of $\Omega_j$. In this way, high values of the standard deviation $\bar{\Omega}$ indicate that the network depicts the groups with distinct mean frequencies. The number of groups was considered $N_{\text{group}} = 10$; however, similar results can be obtained for different numbers of groups $N_{\text{group}}$.

Figure 7 depicts $\bar{\Omega}$ as a function of $p$ and $\varepsilon$. Lower values of the standard deviation (blue tones) are observed for region (1) since this region depicts the nonsynchronized states. A similar scenario is noticed for the phase-synchronized cases as displayed in region (4), where blue tones are observed again. On the other hand, higher values of $\bar{\Omega}$ are observed (green, yellow, and red tones) for regions (2) and (3), which indicate that the groups depict distinct mean frequencies. For these regions, the zig-zag fronts do not show a unique bursting frequency.

![Figure 7](image)

**FIG. 7.** Standard deviation of the bursting frequency ($\bar{\Omega}$) over the $N_{\text{group}} = 10$ groups. Higher values of $\bar{\Omega}$ indicate a coexistence between different frequencies. For regions (1) and (4), the standard deviation depicts lower values (blue tones). For regions (2) and (3), higher values are observed (green, yellow, and red tones).
B. Intermittency and malleability

The intermittency quantifier may be defined by the temporal standard deviation of the time series of $R$ or $\Delta$, which is as follows: \[ \sigma(A) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (A(t) - \langle A \rangle)^2}, \] where $A$ stands for the Kuramoto order parameter or the determinism of the network mean field and $T$ is the size of the time series. $\sigma$ makes the evaluation of the temporal behavior of the synchronization quantifiers possible, where a higher value of $\sigma$ indicates the intermittent behavior. In this sense, it characterizes the variability of the synchronization quantifiers.

In Figs. 8(a) and 8(b), the quantifiers $\sigma(R)$ and $\sigma(\Delta)$ denote the intermittent behavior (higher values) on the transition region [region (3)] for the two cases: the transition induced for the increase of $\varepsilon$ and the transition induced by the rewiring of connections. The first case was explored in Refs. 44, where the existence of nonstationary transitions in small-world and random networks was observed. On the other hand, the intermittency on the transition induced by the increase of $p$ is a new phenomenon. Besides this, region (2) depicts an intermittent behavior, which indicates that the zig-zag fronts are not stable changing time function. For regions (1) and (4), the nonsynchronized and phase-synchronized cases, the intermittent behavior is not observed in both quantifiers [$\sigma(R)$ and $\sigma(\Delta)$]. A similar scenario was already reported in the literature.44

The intermittent synchronization details may be analyzed using the distribution of synchronization quantifiers.57,58 For this particular case, the determinism of the network mean field is more suitable than the Kuramoto order parameter to characterize the intermittent properties.57,44

Figure 9 depicts the probability distribution function (PDF) of the time series obtained by the windowed computed determinism for different values of $p$ and $\varepsilon$ such that different regions observed in the synchronization scenario can be evaluated. Here, a windowed computed determinism time series of size $5 \times 10^7$ points is considered for each case.

Panel (a) is representative of region (2) where $p = 8 \times 10^{-3}$ and $\varepsilon = 0.030$ (black line), $\varepsilon = 0.035$ (cyan line), and $\varepsilon = 0.040$ (magenta line). For these cases, a nonstationary behavior is observed since the PDFs display a large interval of $\Delta$ in a nonbimodal distribution.17 A similar scenario is observed in panel (b), a representative region (3) example, where $p = 2 \times 10^{-2}$, and the coupling values are the same as panel (a). Here, the PDFs show a bimodal distribution, which indicates a possible two-state intermittency, as observed in Refs. 17 and 44. Panel (e) depicts a region (3) example, where the transition is induced by the increase of the coupling strength. Here, $p = 6 \times 10^{-1}$ and $\varepsilon = 0.008$ (black line), $\varepsilon = 0.0085$ (cyan line), and $\varepsilon = 0.009$ (magenta line), and a nonstationary behavior is again observed, however, in comparison with panel (b) where the second peak is less pronounced. These distributions indicate that the system may depict an intermittent phenomenon between two states with different levels of synchronization: a more synchronized state (higher $\Delta$ value) and a less synchronized one (lower $\Delta$ value). A similar scenario is observed in random networks.44 Panels (c), (d), and (f) depict stationary cases representative of region (4) [panels (c) $p = 9 \times 10^{-1}$ and (f) $p = 6 \times 10^{-1}$] where phase-synchronized states are observed (higher values of $\Delta$) and region (1) [panel (d) $p = 6 \times 10^{-1}$] where nonsynchronized states are noticed (lower values of $\Delta$). This analysis corroborates the fact that the zigzag structure region (2) may present intermittent behavior, and it helps to understand the process of transition to phase synchronization induced by the increase of $p$, where the two-state intermittency is observed.

The intermittent behavior of bursting networks has been studied as a function of the coupling strength.74 Here, we have shown the existence of nonstationary behavior in the parameter space $p \times \varepsilon$. In order to investigate the intermittency quantifier as a function of

![FIG. 8. Panels (a) and (b) depict the intermittency quantifier based on the Kuramoto order parameter and the determinism of mean field time series, respectively. The regions depicted are the same as Fig. 4: region (1) represents the nonsynchronized states, region (2) the zig-zag fronts, region (3) the transition states, and region (4) the phase-synchronized ones.](image-url)

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where the rewiring probability \( p \), it is possible to use

\[
\Upsilon(p) = \int \sigma(\varepsilon, p) \, ds,
\]

where \( \sigma \) is described by Eq. (14) and can be evaluated from the Kuramoto order parameter, and the determinism and the interval of coupling strength \( \varepsilon \) considered is depicted in Fig. 4.

Figure 10 depicts the intermittency \( \Upsilon \) as a function of \( p \) evaluated from the Kuramoto order parameter (magenta line) and the determinism of the network mean field (black line). For both cases, for \( p < 1 \times 10^{-2} \), a plateau is observed, which is related to the intermittency in the zig-zag fronts [region (2)]. For the interval of \( 1 \times 10^{-2} < p < 3 \times 10^{-2} \), a maximum is noticed for \( R \) and \( \Delta \). These regions are described as region (3), where the transition is induced by the rewiring connection, and they present a higher level of intermittency in the parameter space studied. Finally, for \( p > 3 \times 10^{-2} \), \( \Upsilon \) decreases to a vanishing value since to these regions, the network depicts phase synchronization and the intermittency behavior is restricted to the transition induced by the increase in coupling.

The region where \( \Upsilon \) depicts a maximum value may be characterized as a malleable region. The simulations have considered 20 different connection matrices for each value of \( p \). Figure 11 depicts the maximum and minimum values of the Kuramoto order parameter as a function of coupling strength \( \varepsilon \). The analyses are performed for the network coupled through connection matrices generated by different seeds with the same value of \( p \). Here, the black line depicts the system for \( p = 8 \times 10^{-3} \), the magenta line \( p = 2 \times 10^{-2} \), and the cyan line \( p = 5 \times 10^{-2} \). In the first case, the mean value of the Kuramoto order parameter is always smaller than 0.50, which indicates that the system is always in a nonsynchronized state.

On the other hand, for intermediate cases depicted in Fig. 11, the network can show an almost phase-synchronized case \( (R) \approx 0.75 \), and for a different connection matrix, the network may present a case of \( (R) < 0.4 \), considering the same set of parameter \( \varepsilon \) and \( p \). This

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**FIG. 9.** PDFs of the windowed computed determinism time series with \( 5 \times 10^2 \) points. Panel (a), \( p = 8 \times 10^{-3} \), is representative of region (2) where nonstationary behavior is observed for different values of \( \varepsilon \). Panel (b), \( p = 2 \times 10^{-2} \), is representative of region (3) where the intermittency of two-states is observed. Panel (e), \( p = 6 \times 10^{-3} \), shows results for region (3) where nonstationary behavior is found. In these cases, the distribution of the synchronization quantifier \( \Delta \) indicates that the network depicts an intermittent behavior between two states of a higher and lower synchronization level. Finally, panels (c) \( (p = 9 \times 10^{-3}) \), (d) \( (p = 6 \times 10^{-3}) \), and (f) \( (p = 6 \times 10^{-3}) \) represent region (4) [(c) and (f)] and region (1), characterizing phase-synchronized and nonsynchronized states, respectively. For these cases, a stationary behavior is always observed.

**FIG. 10.** The quantifier \( \Upsilon \) [Eq. (15)] as a function of \( p \) based on the Kuramoto order parameter (magenta line) and the determinism (black line). The results are given by an average over 20 networks for each value of \( p \), and the filling is its dispersion. For \( p < 1 \times 10^{-2} \), a plateau is observed, and for \( 1 \times 10^{-2} < p < 3 \times 10^{-2} \), a maximum is observed for both curves. Higher values of \( p \) lead to \( \Upsilon \) vanishing values.
Fact demonstrates the malleability of the system to this region, which coincides with the maximum of $\tilde{\gamma}$. Finally, for higher values of $p$, the system loses its malleability, and for all simulations, the results are very similar regarding the phase synchronization, as observed by the cyan line.

VI. CONCLUSIONS

We have studied the dynamical properties regarding phase synchronization and intermittency of a network composed of 1000 chaotic bursting oscillators under local and nonlocal connections. In order to investigate the topology influence, we have considered the Watts-Strogatz networks, where the connection matrix is initially regular with a second neighborhood connection scheme, becoming a small-world and random one as a consequence of the increase of the probability rewiring value $p$.

By using the Kuramoto order parameter and recurrence quantification analysis, we have observed the existence of different regions in the parameter space $p \times \varepsilon$: for smaller values of $p$, the network depicts nonsynchronized states; for higher values of $\varepsilon$ and higher values of $p$, the phase-synchronized states have been observed; for higher values of $\varepsilon$ and smaller values of $p$, diagonal structures and zig-zag fronts have been noticed in the space-time plot. Besides this, this region depicts intermittent behavior, which indicates that the zig-zag structures are not stable. This phenomenon can be understood by the influence of topology since the local connection scheme results in a high value of local efficiency, but a low value of global efficiency, which does not allow the phase synchronization to occur. Moreover, for the region where zig-zag fronts are observed, we have noticed a malleable behavior since, for the same set of parameters $p$ and $\varepsilon$, the network may depict different levels of phase synchronization.

Furthermore, we have observed two distinct transitions to phase synchronization: the first one characterized by the transition from nonsynchronized to phase-synchronized states, which is caused by the increase of coupling strength; the second one consists of a transition from zig-zag fronts to phase-synchronized states, which results in the increase of $p$ and the rewiring of local connections to random ones. In both cases, a nonstationary behavior is observed, where for some cases, the existence of two-state intermittency is noticed.

The results have shown the influence of topology and the complex phenomena observed in the parameter space of $p \times \varepsilon$. Using a discrete map with a simple coupling scheme, it is possible to observe several rich dynamical behaviors, where the rewiring of connections may change deeply the synchronization regime. This approach may offer a theoretical/computational view to real problems since these topology characteristics are observed in neural systems, and synchronization and intermittency play important roles in the functioning of the brain.

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