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Influence of chaotic synchronization on mixing in the phase space of interacting systems

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Using the concept of the relative metric entropy, we study the influence of the synchronization phenomenon on mixing rate in the phase space of deterministic and noisy chaotic systems. We show that transition to both complete and phase synchronization of chaos is accompanied by the decrease of the level of mixing induced by internal nonlinear mechanisms of interacting systems as well as by external noise influence. Therefore, the decrease of the mixing rate in the phase space of interacting systems may indicate transition to synchronization. The obtained results are important for time series analysis in various types of real noisy systems (e.g., biological, social, and financial systems). © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4773824]

Chaotic and stochastic systems are characterized by the mixing phenomenon which leads to the finiteness of time of predictability of the evolution of a cloud of initial points in the phase space. The mixing rate is characterized by the Kolmogorov-Sinai entropy for dynamical systems. In case of a noisy system, the relative metric entropy has been introduced.1 Recently, it has been demonstrated that the phase synchronization of periodic and quasi-periodic oscillations in presence of noise decreases the relative metric entropy.2 In this paper, we show that synchronization of chaos with and without noise is also accompanied by the decrease of the relative metric entropy. This means that the synchronization phenomenon initiates the decrease of the level of mixing caused by internal dynamics of chaotic systems both in presence and without external noise. The presented results enable us to conclude that from one hand any kind of synchronization leads to decrease of the mixing in a system and from the other hand the decrease of the relative metric entropy can indicate the synchronization in a system.

I. INTRODUCTION

Synchronization of chaos is one of the fundamental effects in nonlinear dynamics.3,4 Nowadays, chaotic synchronization is rather well studied.5 Different types of chaotic synchronization have been introduced: complete synchronization,6 phase synchronization,7,8 and generalized synchronization.9 Complete synchronization is the most simple type of chaotic synchronization, and corresponds to the situation when oscillations of partial systems become completely identical. Phase synchronization is a generalization of the synchronization theory for periodic oscillations to the case of the phase-coherent chaos when the Fourier spectrum of chaotic oscillations has a peak on a frequency determined by parameters of system.

Synchronization in both regular and chaotic cases is intuitively understood as a phenomenon which leads to some kind of regularization and consistency of system behavior. In this sense, synchronous oscillations seem to be less complex than unsynchronized ones. There are a lot of publications (e.g., Refs. 10–14) where the idea of using of complexity measures to quantify the synchronization was applied.

One of the most straightforward measures for the complexity of oscillations in chaotic or noisy systems is the rate of mixing. In the case of a deterministic system, the mixing rate is characterized by the Kolmogorov-Sinai entropy.15 When noise is inserted to the system, the relative metric entropy1,2 is used to characterize the mixing introduced by both deterministic mechanism (dynamical chaos) and stochastic forcing (noise). It has been shown that the phase synchronization of periodic and quasi-periodic self-sustained oscillations under the influence of noise leads to decrease of the relative metric entropy.2 Hence, phase synchronization of regular oscillations in presence of noise decreases the rate of mixing introduced by external noise. However, the question of how the synchronization phenomenon affects the mixing rate of chaotic oscillations in interacting systems remains open.

To answer this question, we have considered two models: coupled Rössler oscillators and coupled Anishchenko-Astakhov oscillators both in deterministic case and in presence of external white Gaussian noise. We have estimated the mixing rate by the value of the relative metric entropy. The obtained results show that the transition to complete synchronization as well as to the phase synchronization of chaos initiates the decrease of mixing in the phase space of interacting systems. These results remain under the influence of external white gaussian noise.

The paper has the following structure. In Sec. II, we review the relative metric entropy and explain how it estimates the mixing rate in the case of synchronization of interacting systems. Next, we present the results of the complete
synchronization influence on the mixing rate in considered models. In Sec. IV, we consider the case of phase synchronization. In Conclusion, we outline our results.

II. RELATIVE METRIC ENTROPY

The mixing phenomenon in the phase space of a dynamical system is characterized quantitatively by the Kolmogorov-Sinai metric entropy \(\text{KS-entropy}\).\(^{15}\) Recently a similar characteristic has been introduced for the case of a noisy system—the relative metric entropy.\(^1\) Without noise, this quantity gives an estimate for the value of KS-entropy.

The notion of the relative metric entropy is based on the analysis of the recurrence plots.\(^{16}\) Let a dynamical system be represented by a single trajectory \(\{x_k\}, k = 1, ..., N\) in a \(d\)-dimensional phase space. Then the following matrix can be defined:

\[
R_{kj} = \Theta(\varepsilon - \|x_k - x_j\|), \quad k, l = 1, ..., N,
\]

where \(\varepsilon\) is a certain threshold and \(\Theta\) is the Heavyside function. A graphical representation of \(R_{kj}\) on a discrete coordinate plane \((k, l)\) where a black dot is placed in \((k_0, l_0)\) if \(R_{k_0l_0} = 1\) is called the Recurrence Plot (RP). The quantity obtained from RP

\[
\hat{K}_2(\varepsilon, n) = \frac{1}{n\Delta t} \ln \left( \frac{1}{N^2} \sum_{k,l=1}^{N} \prod_{m=0}^{n-1} R_{k+m,f+m} \right)
\]

is called the relative metric entropy\(^1,2\) as it gives a KS-entropy estimate relatively to the diameter \(\varepsilon\) of the phase space volume which is assumed to be infinitesimal.

It has been shown\(^2\) that the synchronization of periodic and quasi-periodic oscillations in the presence of noise leads to decrease of the relative metric entropy. Hence, synchronization of regular oscillations in presence of noise decreases the level of mixing inserted into the system by external noise. In Secs. III and IV, we consider the influence of chaotic synchronization on the mixing in both presence and without external noise influence.

III. THE INFLUENCE OF COMPLETE SYNCHRONIZATION ON MIXING IN THE PHASE SPACE OF INTERACTING CHAOTIC SYSTEMS

One of the most straightforward definitions of synchronization of chaotic systems is so-called complete synchronization.\(^{5,17,18}\) In case of complete synchronization, the interacting identical chaotic systems denoted by phase variables \(x_1\) and \(x_2\) demonstrate identical oscillations, i.e., \(x_1 = x_2\) at every time moment. We start our study from the case of complete synchronization of coupled Rössler oscillators.

A. Coupled identical Rössler oscillators

The complete synchronization of chaos can be observed in the system of two coupled identical Rössler oscillators:\(^{19}\)

\[
\begin{align*}
x_{1,2} &= -y_{1,2} - z_{1,2} + \gamma(x_{2,1} - x_{1,2}), \\
y_{1,2} &= x_{1,2} + ay_{1,2}, \\
z_{1,2} &= b + z_{1,2}(x_{1,2} - c).
\end{align*}
\]

Here \(x_{1,2}, y_{1,2},\) and \(z_{1,2}\) are the state variables, \(a, b,\) and \(c\) are control parameters, \(\gamma\) is the coupling coefficient. Parameters are chosen to be \(a = b = 0.2, c = 6.5\) in order to produce chaotic dynamics in uncoupled systems. Fig. 1 shows the mean synchronization error defined as follows: \(\langle \varepsilon \rangle = \sqrt{\langle (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \rangle}.

It is clearly seen from Figure 1 that the complete synchronization occurs at \(\gamma > 0.092\).

The relative metric entropy of the system (2) is presented in Fig. 2 versus the coupling coefficient.

One can see that the coupling coefficient raise leads to the decrease of the relative metric entropy \(\hat{K}_2\) until \(\gamma\) crosses a certain value around 0.05 where the relative entropy reaches plateau. The further increase of the coupling coefficient leads to small fluctuations of \(\hat{K}_2\) in vicinity of an average value which is close to the one for a single Rössler oscillator with the same control parameters.\(^1\) However, the plateau starts earlier then the complete synchronization appears. The rate of mixing is a statistical characteristics, hence to find out the reasons, we consider a two-dimensional

FIG. 1. The mean synchronization error \(\langle \varepsilon \rangle\) versus the coupling coefficient \(\gamma\) for the system (2) with \(a = b = 0.2, c = 6.5\). For \(\gamma > 0.092\), complete synchronization is observed in the system. Insets demonstrate the projections of the phase space of the system (2) to \((x_1, x_2)\) plane.

FIG. 2. Relative metric entropy of the system (2) with \(a = b = 0.2, c = 6.5\) versus the coupling coefficient. Complete synchronization is observed for \(\gamma > 0.092\).
probability density distribution (2D PDD) in $(x_1, x_2)$ plane for two values of the coupling coefficient: $\gamma = 0.01$ (before the plato) and $\gamma = 0.05$ (on the plato before the complete synchronization starts). The results are presented in Fig. 3.

As one can see, the complete synchronization is preceded by the formation of a ridge in 2D PDD along the line $x_1 = x_2$ due to the mechanism of complete synchronization of chaos described in Ref. 4. It is emergence of the ridge in 2D PDD leads to the decrease of mixing rate before the complete synchronization is established.

The presented results for the model system (2) enable us to conclude that the transition to the regime of complete synchronization is accompanied by the reduction of the relative metric entropy and therefore by the decrease of the mixing rate.

For a real-world system, it is a common situation when the external noise is present. Hence, the following question arises: does the obtained result remain in the presence of the noise? We have done the same numerical simulations with addition of white Gaussian noise to the system:

$$\begin{align*}
\dot{x}_1 &= -y_1 - z_1 + \gamma(x_2 - x_1) + \sqrt{2}Dn(t), \\
\dot{y}_1 &= x_1 + ay_1, \\
\dot{z}_1 &= b + z_1(x_1 - c), \\
\dot{x}_2 &= -y_2 - z_2 + \gamma(x_1 - x_2), \\
\dot{y}_2 &= x_2 + ay_2, \\
\dot{z}_2 &= b + z_2(x_2 - c).
\end{align*}$$

(3)

Here $n(t)$ is white Gaussian noise source, $D$ is the noise intensity. The parameters are $a = b = 0.2$, $c = 6.5$. The results are presented in Fig. 4 for three different values of the noise intensity $D$. As one can see, the effect observed in deterministic system remains in the presence of the white Gaussian noise.

**B. Coupled identical Anishchenko-Astakhov oscillators**

Now, let us consider another well-known model system—the Anishchenko-Astakhov oscillator under the influence of white Gaussian noise:

$$\begin{align*}
\dot{x}_1 &= (m - z_1)x_1 + y_1 + \gamma(x_2 - x_1) + \sqrt{2}Dn(t), \\
\dot{y}_1 &= -x_1, \\
\dot{z}_1 &= g(f(x_1) - z_1), \\
\dot{x}_2 &= (m - z_2)x_2 + y_2 + \gamma(x_1 - x_2), \\
\dot{y}_2 &= -x_2, \\
\dot{z}_2 &= g(f(x_2) - z_2).
\end{align*}$$

(4)

Here $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$ are the state variables of the system, $m$ and $g$ are the control parameters, $f(x)$ is 0 for $x < 0$ and $x^2$ for $x \geq 0$, $\gamma$ is the coupling coefficient, $n(t)$ is white Gaussian noise source, $D$ is the noise intensity. We have chosen the parameters of the system to be $m = 1.42$, $g = 0.2$ to produce chaotic dynamics in uncoupled systems. Without noise ($D = 0$) the system possesses complete synchronization at $\gamma \geq 0.125$.

The relative metric entropy versus the coupling coefficient is shown in Fig. 5 (without noise) and in Fig. 6 (in the presence of noise). There is an interval of $\gamma$ where the relative metric entropy is close to zero due to the qualitative changes in the dynamics of interacting systems which are clearly seen in the Figure 5. Nevertheless, when the complete synchronization is established, the interacting systems return into the chaotic regime which is characterized by the lower value of the mixing rate than for $\gamma \rightarrow 0$.

**IV. PHASE SYNCHRONIZATION INFLUENCE ON MIXING IN THE PHASE SPACE OF INTERACTING SYSTEMS**

In order to study phase synchronization, a few approaches have been proposed to calculate phases of chaotic oscillators. In general, one can apply the analytic signal approach introduced by Gabor.

**A. Phase synchronization of coupled chaotic Rössler oscillators**

At first, we consider two coupled chaotic Rössler oscillators\textsuperscript{17}

![FIG. 4. Relative metric entropy versus the coupling strength for the system (3) with $a = b = 0.2$, $c = 6.5$, and the different values of the noise intensity $D$. Without noise, complete synchronization of chaos is observed for $\gamma > 0.092$.](Image)
The projections of the phase space of the system (4) to chaos is observed for $c > 0$. Without noise, complete synchronization of chaos is observed for $c > 0$. Keeping in mind that phase synchronization appears for $\gamma > 0.036$, the results presented in Fig. 7 enable us to conclude that the phase synchronization phenomenon decreases the mixing rate in the phase space of deterministic chaotic system.

Keeping in mind that phase synchronization appears for $\gamma > 0.036$, the results presented in Fig. 7 enable us to conclude that the phase synchronization phenomenon decreases the mixing rate in the phase space of deterministic chaotic system.

The local minimum of $\hat{K}_2$ near the bifurcational value of the coupling coefficient is due to the change in the character of the oscillatory regime of the system in the vicinity of $\gamma = 0.038$. Here, the spectrum of the first oscillator is characterized by a set of well pronounced peaks. Hence, the system produces more regular oscillations leading to the decrease of the relative metric entropy of the entire system.

Now we confirm that the effect presented in Fig. 7 remains for a noisy system. The system of two coupled Rössler oscillators with white Gaussian noise is represented by the following equations:

$$
\begin{align*}
\dot{x}_{1,2} &= -\omega_{1,2}x_{1,2} - z_{1,2} + \gamma(x_{2,1} - x_{1,2}), \\
\dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + ay_{1,2}, \\
\dot{z}_{1,2} &= b + z_{1,2}(x_{1,2} - c),
\end{align*}
$$

with a small parameter mismatch: $\omega_{1,2} = 0.97 \pm \Delta \omega$ and $a = 0.165$, $b = 0.2$, and $c = 10$. Both oscillators possess coherent phase dynamics due to the rotation with a small variation of the return time, but they have different average frequencies as a result of the mismatch $\Delta \omega$.

Figure 8 presents relative metric entropy for the system (6) versus coupling coefficient for different values of the noise intensity. As one can see, the effect of the decrease of
The relative metric entropy is valid for a system with noise. The local minimum observed in Fig. 7 vanishes, hence the corresponding regime is non-robust.

B. Phase synchronization in coupled Anishchenko-Astakhov oscillators

We have carried out a series of similar numerical simulations for another well-known model system—coupled Anishchenko-Astakhov oscillators

\[
\begin{align*}
\dot{x}_1 &= (m - z_1)x_1 + y_1 + \gamma(x_2 - x_1) + \sqrt{2Dn(t)}, \\
\dot{y}_1 &= -x_1, \\
\dot{z}_1 &= g(f(x_1) - z_1), \\
\dot{x}_2/p &= (m - z_2)x_2 + y_2 + \gamma(x_1 - x_2), \\
\dot{y}_2/p &= -x_2, \\
\dot{z}_2/p &= g(f(x_2) - z_2).
\end{align*}
\] (7)

Here \( p \) is the frequency detuning between two oscillators, \( n(t) \) is white Gaussian noise source, \( D \) is the noise intensity. The parameters are \( m = 1.42, g = 0.2 \) and the frequency detuning is \( p = 1.05 \). With the chosen parameters phase synchronization occurs at \( \gamma = 0.019^4 \).

The value of the relative metric entropy for the system (7) versus the coupling coefficient is presented in Fig. 9 without noise and in Fig. 10 for different noise intensities. The raise of the relative metric entropy before phase synchronization is established is due to the abrupt transition to a coexisting chaotic regime with a higher mixing rate and small basin of attraction. When noise is inserted into the system, the phase trajectory does not remain on this attractor and the relative metric entropy raise disappears. As one can see, the mixing decreases in phase synchronization regime in both considered cases.

V. CONCLUSION

In this paper, we have studied the influence of synchronization phenomenon on mixing in the phase space of interacting chaotic systems. Two types of chaotic synchronization were considered: complete synchronization and phase synchronization. Using two basic model chaotic systems, we have shown that synchronization of chaos is accompanied by a decrease of the relative metric entropy (RME) after a certain coupling coefficient value can indicate transition to synchronization in both regular systems under a noise influence and in chaotic systems (deterministic as well as noisy).

This fundamental result can be applied in experimental studies of synchronization phenomena. To evaluate the synchronization of coupled systems using time series, one has to calculate \( K_2 \) for different coupling coefficient values. The decrease of the relative metric entropy (RME) after a certain coupling coefficient value can indicate transition to synchronization. This method is universal for different types of oscillations and allows using of short time series (in the present examples we have used 10 000 points per series, however, this number can be reduced). This can be crucial for a lot of real-world systems like biological or climatic systems. However, despite the universality of such indicator, the qualitative changes in dynamical regimes of interacting systems must be taken into account. For example, if the route to synchronization of chaos includes transition into a regime with lower mixing rate (periodic or more coherent chaos) then the decrease of the relative metric entropy can be observed without synchronization. Therefore, this method can be used only for a primary analysis and the obtained results must be verified because the method is highly sensitive to the changes in statistical properties of a system’s regime.

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