Complex network analysis of recurrences in phase space

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Jonathan Donges, Reik Donner, Yong Zou, Jürgen Kurths
Outline

1. Recurrences
2. Recurrence plots
   • definition, structures, quantification, examples
3. Network analysis of recurrences
   • definition, network measures, clustering, examples
Recurrences
Recurrence

- fundamental characteristic of many dynamical systems
- recurrences in real life:
  Milankovich cycles, weather after storm, El Niño phenomenon, heart beat after exertion, Maya calendar etc.
Recurrence

- Anaxagoras, approx. 450 BC:
  perichoresis: chaotic circular movement
Recurrence

- Poincaré, 1890:

  “a system recurs infinitely many times as close as one wishes to its initial state”
Investigating Recurrence

- Poincaré map
- Recurrence time statistics
- First return map
- Recurrence plot
- Network analysis of recurrences
Recurrence Plots
Recurrence Plot

Recurrence Plot

Recurrence Plot

Recurrence Plot

Recurrence Plot

\[ R_{t_1,t_2} = \Theta (\varepsilon - \| \vec{x}(t_1) - \vec{x}(t_2) \|) \]

Euclidean Norm

Small-scale texture is visible in fig. 2 in the form of short lines parallel to the diagonal of the recurrence plot. Such lines would also be visible in fig. 1 and 4 at a larger magnification: they correspond to sequences \((i, j), (i+1, j+1), \ldots, (i+k, j+k)\) such that the piece of trajectory \(x(j), x(j+1), \ldots, x(j+k)\) is close to \(x(i), x(i+1), \ldots, x(i+k)\). The length of the lines is thus related to the inverse of the largest positive Liapunov exponent.

If the \(x(i)\) were randomly chosen rather than coming from a dynamical system, there would be no such lines. Another type of texture, the checkerboard texture, is expected for the Lorenz system, corresponding to the fact that \(x(i)\) moves on a spiral sometimes around one, sometimes around the other of the two symmetric fixed points of the system. The checkerboard structure, however, is only barely visible in fig. 2, but a careful analysis shows that the diagonal black spots fall into two mutually exclusive groups having again black spots on the intersection of the horizontal and vertical lines. In fact, the striking events visible in fig. 2 are those when \(x(i)\) spirals for a long time around one of the fixed points. The pattern thus created extends to large scales, and can no longer quite be called a texture.

To conclude, we wish to stress that the recurrence plots are rather easily obtained aids for the diagnosis of dynamical systems. They display important and easily interpretable information about time scales which are otherwise rather inaccessible.

Eckmann et al, EPL, 1987:
Recurrence Plot

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Eckmann et al, EPJST, 2008:

paper. It is of course rewarding to discover that a small paper has, after a dormant period, led to an active field, with many ramifications we certainly had not anticipated. One can wonder what
Recurrence Plot Publications

- Introduction
- Quantification
- Time-dependent quantification
- Reconstruction of driving force
- Cross-RP
- Joint-RP & synchronisation
- Dynamical invariants
- Surrogates
- Coupling direction
- Recurrence networks
- Deterministic chaos
- Gap filling

Publication counts:
- 1987: 1
- 1989: 1
- 1991: 1
- 1993: 2
- 1995: 3
- 1997: 11
- 1999: 17
- 2001: 20
- 2003: 27
- 2005: 36
- 2007: 82
- 2009: 115
- 2011: 127
- 2017: 108
- 2019: 107
- 2021: 22

Publications
Recurrence Plot

- to visualise the phase space trajectory by its recurrences

- recurrence matrix:
  - binary
  - symmetric

\[
R_{i,j} = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

N. Marwan et al., Physics Reports, 438, 2007
Recurrence Plot Typology

- Homogeneous
- Periodic
- Drifty
- Disrupted
Recurrence Quantification

- number of lines of exactly length \( l \)
  - histogram \( P(l) \)

Recurrence Quantification

- Recurrence rate

\[ RR = \frac{1}{N^2} \sum_{i,j=1}^{N} R_{i,j} \]

Probability that any state recurs

- Determinism

\[ DET = \frac{\sum_{l=l_{\text{min}}}^{N} l P(l)}{\sum_{l=1}^{N} l P(l)} \]

Probability that recurrences further recur
Time Depending Analysis

- sliding window: detection of dynamical transitions
Dynamics of Oxygen Crises in a Lake

Facchini et al., Ecological Modelling, 203, 2007
Recurrence Plot

- Transition detection
- Differentiate dynamics
- Finding time scales
- Interrelation detection
- Synchronisation analysis
- Surrogates
- Recurrence time statistics
- etc.

N. Marwan et al., Physics Reports, 438, 2007
Complex Networks
Complex Networks

- Link matrix (undirected, unweighted network):
  - Binary
  - Symmetric

\[ A_{i,j} = \]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\]
Complex Networks

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  ▶ binary
  ▶ symmetric

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1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
\end{bmatrix} \]

▶ link matrix: similar to recurrence plot
Time Series Analysis using Complex Networks

- Link matrix = recurrence matrix of time series
  - Nodes: states in phase space
  - Links: local neighbours of states (i.e. recurrence)

- Path: connected neighbourhoods

Time Series Analysis using Complex Networks

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Time Series Analysis using Complex Networks

- Complex network measures applied to recurrence plot
  - measures of complexity explaining topological properties of complex systems
  - local and global measures
- "recurrence network"

Donner et al., IJBC 21, 2011
<table>
<thead>
<tr>
<th>Scale</th>
<th>Network measure</th>
<th>Phase space</th>
</tr>
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<tbody>
<tr>
<td>Local</td>
<td>link density</td>
<td>global recurrence rate</td>
</tr>
<tr>
<td></td>
<td>degree centrality</td>
<td>local recurrence rate</td>
</tr>
<tr>
<td>Intermediate</td>
<td>clustering coefficient</td>
<td>invariant objects, local dimension</td>
</tr>
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<td>local degree anomaly</td>
<td>local heterogeneity of phase space density</td>
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<td></td>
<td>matching index</td>
<td>twinness</td>
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<tr>
<td>Global</td>
<td>average path length</td>
<td>mean phase space separation</td>
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<td></td>
<td>network diameter</td>
<td>phase space diameter</td>
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<td></td>
<td>global transitivity/ clustering</td>
<td>regular dynamics</td>
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<td></td>
<td>motif distribution</td>
<td>dynamical classification</td>
</tr>
</tbody>
</table>
Clustering Coefficient
Clustering Coefficient
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Clustering Coefficient

- probability that neighbours of a node are also connected
Clustering Coefficient

\[ C_v = \frac{\sum_{i,j} A_{v,i} A_{i,j} A_{j,v}}{k_v (k_v - 1)} \]

- probability that neighbours of a node are also connected
Clustering Coefficient in Phase Space

Regular/periodic

\[ C \approx 1 \]

Diverging/chaotic

\[ C < 1 \]
Clustering Coefficient in Phase Space

- Regular/periodic: $C \approx 1$
- Diverging/chaotic: $C < 1$

- Clustering coefficient: regularity of dynamics, system dimension
Example: Logistic Map

- Logistic map:
  \[ x_{i+1} = a x_i (1 - x_i) \]
Example: Logistic Map
Example: Logistic Map

- Control parameter $a$

- Periodic windows

- Graph showing bifurcation diagram and period-doubling route to chaos

- Critical values for period doubling: $a_1 = 3.571, a_2 = 3.830, a_3 = 3.851, a_4 = 3.858, a_5 = 3.860$
Example: Logistic Map
Example: Logistic Map

![Logistic Map Diagram]

- Clustering coefficient
- Control parameter $a$

› intermittent phases
Example: Logistic Map
Asian Monsoon

Indian Monsoon

East Asian Monsoon

Qunf

Krum Umsynrang

Dongge
Monsoon
Asian Monsoon
Asian Monsoon
Asian Monsoon

![Diagram of Asian Monsoon with δ¹⁸O values over time in years before present (yr BP).]
Asian Monsoon

Graph showing the variation of $\delta^{18}O$ with age (yr BP) from 0 to 11,000. The graph indicates periods of wet and dry conditions.
Asian Monsoon

![Graph showing δ¹⁸O values over time (yr BP). The x-axis represents age in years before present (yr BP), ranging from 0 to 11,000. The y-axis represents δ¹⁸O values, ranging from -7 to -4. The graph shows a general trend of decreasing δ¹⁸O values over time.]
Asian Monsoon

\[\delta^{18}O\]

Age (yr BP)

-7 -6 -5 -4

0 1000 2000 3000 4000 5000 6000 7000 8000 9000 10000 11000
Asian Monsoon

Indian Monsoon

East Asian Monsoon

Krum Umsynrang

Dongge

Qunf
3900-3700 yr BP
Harappan culture vanished
Summary

• Complex networks from time series
• Identification and classification of dynamics (regular – chaotic)
• Detection of transitions in dynamics (bifurcations, structural discontinuities)
• Complementary analysis to traditional recurrence analysis
Alternative Approaches

• Visibility graph
• Cycle network
• Correlation network
• Transition network
Key Publications


