Amplitude Equations from Spatiotemporal Binary-Fluid Convection Data

Henning U. Voss*
Institut für Physik, Universität Potsdam, 14469 Potsdam, Germany

Paul Kolodner
Bell Laboratories, Lucent Technologies, Inc., Murray Hill, New Jersey 07974-0636

Markus Abel and Jürgen Kurths
Institut für Physik, Universität Potsdam, 14469 Potsdam, Germany
(Received 29 January 1999)

We apply a recently developed method for the analysis of spatiotemporal data to extract the dynamical equations that describe an experiment on traveling-wave convection in a binary fluid. The technique is based on nonlinear regression analysis and allows the nonparametric estimation of the functions involved in these equations. We find that the system is well described by a pair of coupled complex Ginzburg-Landau equations, and the coefficient of the term that describes the interaction between the two oppositely propagating waves can be determined.

PACS numbers: 47.52.+j, 05.45.–a, 07.05.Kf, 47.20.Ky

A fundamental problem in the study of spatially extended dynamical systems is the quantitative comparison of experimental data with models based on partial differential equations. In the study of nonlinear pattern-forming systems, theoretical models usually take the form of amplitude equations whose derivation is based on symmetry considerations and separation-of-scales arguments [1]. In many cases, comparisons between data and such models are only qualitative. In the case of Rayleigh-Bénard convection in pure fluids, a first-principles derivation of the amplitude-equation model from the Navier-Stokes equations, and quantitative comparison of that model with experimental data, were performed long ago [2]. In the dynamically richer case of traveling-wave convection in binary fluids, the complex Ginzburg-Landau equation (CGLE) model has also been derived directly from the Navier-Stokes equations [3]. (For a review of the literature on this system, see [4].) Given the quality of the experimental data available in this system, a quantitative comparison of this model with data is also warranted.

Here, we apply a recently proposed method [5,6] based on nonparametric nonlinear regression analysis, for the analysis of a simple dynamical state of convective traveling waves. This analysis allows the estimation of the coefficients in the CGLE that yield a best fit to the experimental data. Furthermore, this nonparametric approach allows the estimation of the functions appearing in the CGLE, including the nonlinear terms that describe the coupling between oppositely propagating waves. The agreement between the data and the results of this analysis can be taken as strong evidence for the quantitative validity of the CGLE model for the description of binary-fluid convection near onset.

We analyze data from an experiment on convection in an ethanol/water mixture in a long, narrow, annular container which is heated from below. The system can be considered approximately one dimensional, with periodic boundary conditions. The bifurcation parameter ε is defined as the fractional distance above the temperature difference ΔT_{co} applied across the fluid layer at onset, i.e., $\varepsilon = (\Delta T - \Delta T_{\mathrm{co}})/\Delta T_{\mathrm{co}}$. The convection pattern is visualized by a shadowgraph system which illuminates a circular array of photodiodes, whose signals are digitized to provide data for analysis.

The first dynamical state observed above onset ($\varepsilon = 0$) consists of pairs of weakly nonlinear wave packets which propagate around the system in opposite directions, referred to as "left" and "right" [7] (see Fig. 1). The left- and right-going complex wave amplitudes $A_L(x,t)$ and $A_R(x,t)$ are extracted from the shadowgraph data using complex demodulation [8]. This technique consists, in essence, of Fourier transforming the data in time and space, isolating the peaks corresponding to the left- and right-going waves, and transforming these components separately back into real space, so that their complex amplitudes can be independently measured.

The actual data fields we analyze are the real amplitudes and phases $a_L(x,t)$, $a_R(x,t)$, $\phi_L(x,t)$, and $\phi_R(x,t)$, defined by

$$A_{L,R}(x,t) = a_{L,R}(x,t)e^{i\phi_{L,R}(x,t)}. \quad (1)$$

These fields are sampled on a spacetime mesh of 180 spatial points by 760 time steps, covering 8 to 10 round trips of the wave packets with a temporal resolution of several samples per oscillation cycle. We analyze data obtained at the following seven values of the bifurcation parameter (scaled by the characteristic time τ_0 defined below): $\varepsilon\tau_0^{-1} = 1.77, 4.22, 6.38, 9.32, 12.07, 14.03$, and 16.28×10^{-3}. For each value of $\varepsilon\tau_0^{-1}$, the left- and right-wave fields are analyzed separately.

 0031-9007/99/83(17)/3422(4)$15.00

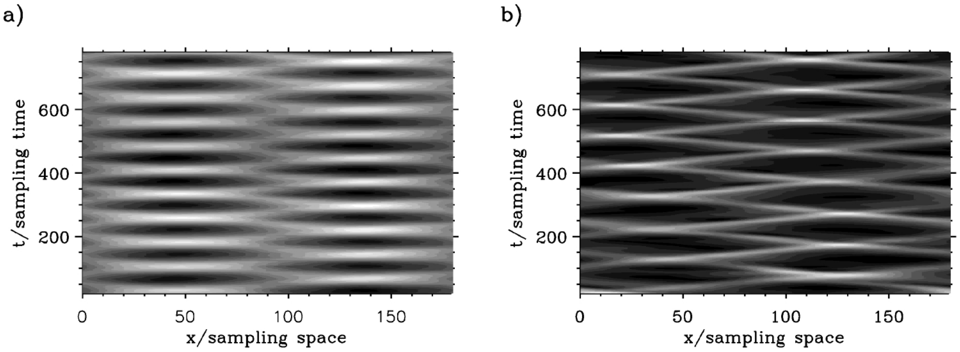

FIG. 1. Space-time plots of the sum of the left- and right-going real amplitudes, given as gray values (small values, dark) for (a) $\varepsilon\tau_0^{-1} = 1.77 \times 10^{-3}$ and (b) $\varepsilon\tau_0^{-1} = 12.07 \times 10^{-3}$. The gray-scale contrast in (a) has been magnified by a factor of 2.5 with respect to that in (b) to compensate for weaker amplitudes.

For the first dynamical states above onset, theory proposes as a quantitative description by amplitude equations two coupled CGLEs for left- and right-going traveling waves [7]. For ease of notation, we consider only right-going waves coupled to left-going waves and suppress the "R" subscript. The equations for left-going waves follow symmetrically with a change of the sign of the velocity s from negative to positive. The complex amplitude $A(x,t)$ is described by the CGLE

$$\tau_0(\partial_t + s\partial_x)A = \varepsilon(1 + ic_0)A + \xi_0^2(1 + ic_1)\partial_{xx}A + g(1 + ic_2)|A|^2A + h(1 + ic_3)|A_L|^2A. \tag{2}$$

The parameter ξ_0 is a characteristic length scale, and τ_0 is a characteristic time which is determined experimentally by measuring the growth rate $\gamma = \varepsilon\tau_0^{-1}$ at several values of ε and fitting the slope [4]. s is the linear group velocity, c_0–c_3 are dispersion coefficients, g is a nonlinear saturation parameter, and h is a nonlinear coupling coefficient which reflects the stabilizing interaction between oppositely propagating traveling-wave components [9]. Inserting Eq. (1), two equations for real amplitude and phase are obtained,

$$\begin{aligned}\partial_t a = {} & -s\partial_x a + \varepsilon\tau_0^{-1}a + \xi_0^2\tau_0^{-1} \\ & \times \{\partial_{xx}a - a(\partial_x\phi)^2 - c_1(2\partial_x a\partial_x\phi + a\partial_{xx}\phi)\} \\ & + g\tau_0^{-1}a^3 + h\tau_0^{-1}a_L^2 a\,,\end{aligned} \tag{3}$$

$$\begin{aligned}a\partial_t\phi = {} & -sa\partial_x\phi + \varepsilon c_0\tau_0^{-1}a + \xi_0^2\tau_0^{-1} \\ & \times \{2\partial_x a\partial_x\phi + a\partial_{xx}\phi + c_1[\partial_{xx}a - a(\partial_x\phi)^2]\} \\ & + gc_2\tau_0^{-1}a^3 + hc_3\tau_0^{-1}a_{\mathrm{L}}^2 a\,.\end{aligned} \tag{4}$$

The main purpose of this Letter is to extract the governing amplitude equations from the experimentally measured data fields. Following the analysis method described in [5], we fit the amplitude equations in a nonparametric way to the data, using a truly nonlinear estimation procedure. Thus, our approach differs from other recently published linear parametric approaches applied to similar problems [10,11]. We note that methods based on the phase space reconstruction principle [12] are not directly applicable for extracting amplitude equations from spatiotemporal data.

The method consists of two steps: (1) Estimation of all derivative fields $\partial_t a$, $\partial_x a$, $\partial_x\phi$, etc., that appear in Eqs. (3) and (4) from the experimental measurements of $a(x,t)$ and $\phi(x,t)$. Here we use spectral estimators. Estimation by a simple differencing scheme [5] yields only slightly cruder results. (2) Performance of a nonparametric nonlinear regression analysis of all terms estimated in the first step.

Since we can apply the method to each of the Eqs. (3) and (4) independently, we use two sets (I and II) of variables $v_i (i = 0,\dots,5)$ as input for the regression analysis. Set I, corresponding to Eq. (3): $v_0 = \partial_t a$, $v_1 = \partial_x a$, $v_2 = a$, $v_3 = \partial_{xx}a - a(\partial_x\phi)^2$, $v_4 = 2\partial_x a\partial_x\phi + a\partial_{xx}\phi$, and $v_5 = a_L^2 a$. Set II, corresponding to Eq. (4): $v_0 = a\partial_t\phi$, $v_1 = a\partial_x\phi$, $v_2 = a$, $v_3 = 2\partial_x a\partial_x\phi + a\partial_{xx}\phi$, $v_4 = \partial_{xx}a - a(\partial_x\phi)^2$, and $v_5 = a_L^2 a$.

The nonlinear regression analysis is performed by estimating *optimal transformations* $\Phi_0(v_0)$ to $\Phi_K(v_K)$ (here $K = 5$) that maximize the linear correlation coefficient R of the transformed terms,

$$\Psi(v_0,\dots,v_K) := \left| R\left(\Phi_0(v_0), \sum_{i=1}^{K}\Phi_i(v_i)\right)\right| \overset{!}{=} \max. \tag{5}$$

The functions Φ_i are varied in the space of all measurable functions until the maximum is achieved. We calculate the Φ_i from the data sets I and II numerically by using the ACE algorithm [13]. The resulting optimal transformations are *nonparametric function estimators* and constitute the main result of the analysis. The quality of the fit with optimal transformations is measured by the *maximal correlation* $\Psi(v_0,\dots,v_K)$ [14]: A value of Ψ close to unity means that the terms v_0 to v_K possess a strong (nonlinear) dependence. In case of a linear dependence, the optimal transformations are linear functions, too.

Numerical studies on several dynamical model equations [5] revealed that the CGLE could be estimated with high accuracy from noise-free data, leading to a maximal correlation of almost unity.

As a first question, we want to check that the spatiotemporal evolution of the system can be described by the coupled CGLEs (3),(4). In this case, one expects the following optimal transformations: The function Φ_0 should be the identity, Φ_2 should be a third-order polynomial in a, and all the other functions should be linear, with slopes corresponding to the coefficients in Eqs. (3) and (4). As a check, below we will compare our results with experimentally obtained coefficients from the same experiment as presented in [4]. There it was also shown that most of the experimental values agree reasonably well with the ones calculated from first principles [3]. These experimental values are represented as smooth curves in Fig. 2. Since the polynomials $\varepsilon\tau_0^{-1}a + g\tau_0^{-1}a^3$ and $\varepsilon c_0\tau_0^{-1}a + gc_2\tau_0^{-1}a^3$ have large uncertainties, curves representing their extremal values are shown in the upper and lower panels for Φ_2, respectively. The distributions of the amplitudes, phases, and derivatives are rather inhomogeneous with heavy tails. Therefore, in Fig. 2 the range on the abscissa that is covered by 96% of the data values is marked by vertical dotted lines. Since the optimal transformations are harder to estimate for very sparse data, we consider each 2% of the transformed data values at the edges as outliers.

For the seven analyzed data sets we obtain the following results (Fig. 2):

For large bifurcation parameters ($\varepsilon\tau_0^{-1} \geq 12.07 \times 10^{-3}$), the expected functions coincide quite well with the coefficients found in [4]. In particular, set I (top row of Fig. 2): The estimate for the left-hand side, Φ_0, turns out to be approximately the identity; the estimate for Φ_1 is an approximately linear function in $\partial_x a$ with a slope in good agreement with the wave velocity s measured in [4]; the estimate for Φ_2 can be described by a cubic polynomial in a; the estimates for Φ_3 and Φ_4 are approximately linear, also with correct slopes. The estimate for the coupling term, Φ_5, appears to be approximately linear in $a_L^2 a$ with a clearly negative coupling coefficient.

Set II (second row of Fig. 2) yields similar results, but obviously the estimates for Φ_3 and Φ_4 are worse. The reason is the following: Because of the law of error propagation, the error of the amplitude $a = \sqrt{(\mathrm{Re}A)^2 + (\mathrm{Im}A)^2}$ is of the order of the error of the measurement A, but the error of the phase $\phi = \arctan \frac{\mathrm{Im}A}{\mathrm{Re}A}$ depends on A in such a way that for the particular data considered here it is on average 2 orders of magnitude larger than the error of the amplitude. This leads to worse estimates for the derivatives and, consequently, to worse results for the optimal transformations

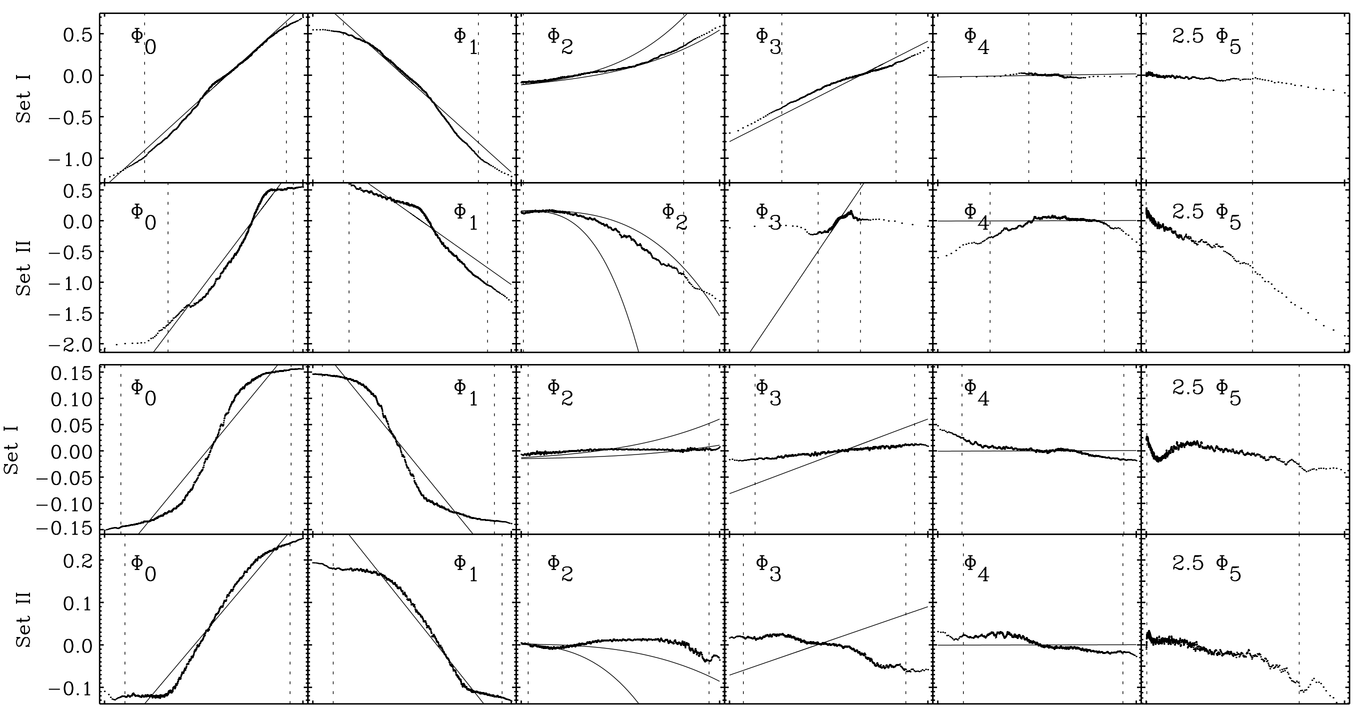

FIG. 2. Estimated optimal transformations for the set of terms I and II, both for $\varepsilon\tau_0^{-1} = 12.07 \times 10^{-3}$ (upper two rows) and for $\varepsilon\tau_0^{-1} = 1.77 \times 10^{-3}$ (lower two rows). The ordinates are the optimal transformations multiplied by 1000. They are the same for all plots in one row, except in the frames for Φ_5 where they have been magnified by 2.5. The abscissa are given by the terms v_0 to v_5, respectively, and are not labeled for clarity. Additionally, smooth curves indicate the theoretically expected functions, and vertical dotted lines mark the range on the abscissa where 96% of the data values are located, as explained in the text. The results for $\varepsilon\tau_0^{-1} = 14.03$ and 16.28×10^{-3} resemble the results for $\varepsilon\tau_0^{-1} = 12.07 \times 10^{-3}$ and are therefore not shown; similarly, the results for $\varepsilon\tau_0^{-1} = 4.22, 6.38$, and 9.32×10^{-3} resemble the results for $\varepsilon\tau_0^{-1} = 1.77 \times 10^{-3}$.

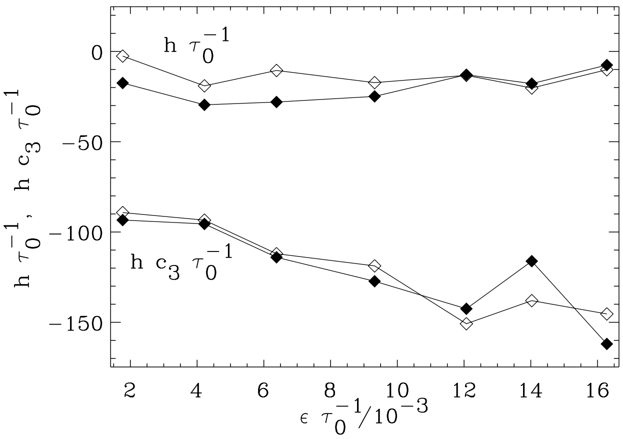

FIG. 3. The estimated coupling coefficients $h\tau_0^{-1}$ and $hc_3\tau_0^{-1}$ for all seven bifurcation parameters. Open and solid symbols denote estimates from left- and right-going waves, respectively.

of the set of terms II. (Note especially that Φ_3 depends on second-order derivatives of the phase.) This is also revealed in a smaller maximal correlation (5): $\Psi(\mathrm{I}) = 0.985$ vs $\Psi(\mathrm{II}) = 0.945$.

For smaller bifurcation parameters ($\varepsilon\tau_0^{-1} \leq 9.32 \times 10^{-3}$), the results are much worse (bottom rows of Fig. 2) and give only a rough impression of the true equations. Here, due to smaller complex amplitudes, the signal-to-noise ratio, given by the ratio of the standard deviations of signal and noise, changes by a factor of 2.7 as $\varepsilon\tau_0^{-1}$ is changed from 1.77×10^{-3} to 16.28×10^{-3}. This leads again to bad estimates of the derivatives for small bifurcation parameters.

Finally, for the three data sets with highest signal-to-noise ratio, we clearly find that the coupling term is approximately of the form $h(1 + ic_3)|A_L|^2A$. This allows one to fit linear functions to the optimal transformations Φ_5 to yield estimates for the coefficients h and hc_3 (Fig. 3). Averaging the results for the three data sets with highest signal-to-noise ratio, we estimate the coupling coefficients to be $h\tau_0^{-1} = -13.7 \pm 4.7$ and $hc_3\tau_0^{-1} = -142 \pm 15$. Note that, as far as we are aware, the coefficients of the coupling term could not be determined with other methods, so we present them for the first time. The negativity of the coupling coefficients indicates a stabilizing nonlinear competition between the two traveling-wave components and a slowing down of the traveling-wave group velocity. This is in agreement with the experimentally observed slight decrease in phase and group velocities measured during the interaction of the oppositely propagating wave packets [7].

To summarize our results, we have shown from an analysis of the dynamics of a state of traveling-wave convection that it is possible to extract the governing amplitude equations from high-quality experimental data. Limitations are mainly due to inaccurate estimates of spatial and temporal derivatives. In our example, this is the case for data just at the onset of convection, where the signal-to-noise ratio is lower than for well-developed amplitudes at higher bifurcation parameters. We expect this approach to be useful for testing and deriving models of spatiotemporal dynamics, using experimental data, and for a quantitative detection of deviations from theory. In future work we will apply the method to investigate the transition to dispersive chaos for higher bifurcation parameters, which is not as well understood as the counterpropagating wave packet regime.

We acknowledge stimulating discussions with G. Flätgen. H. U. V. and M. A. acknowledge financial support from the Max-Planck-Gesellschaft, and H. U. V. and J. K. acknowledge support from the SFB 555 Complex Nonlinear Processes.

*Email address: hv@agnld.uni-potsdam.de

[1] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[2] J. E. Wesfreid *et al.*, J. Phys. (Paris) **39**, 725 (1979).

[3] W. Schöpf and W. Zimmermann, Europhys. Lett. **8**, 41 (1989).

[4] P. Kolodner, S. Slimani, N. Aubry, and R. Lima, Physica (Amsterdam) **85D**, 165 (1995).

[5] H. Voss, M. J. Bünner, and M. Abel, Phys. Rev. E **57**, 2820 (1998).

[6] H. Voss and J. Kurths, Phys. Lett. A **234**, 336 (1997).

[7] P. Kolodner, Phys. Rev. Lett. **69**, 2519 (1992).

[8] P. Kolodner and H. Williams, in *Proceedings of the NATO Advanced Research Workshop on Nonlinear Evolution of Spatio-Temporal Structures in Dissipative Continuous Systems,* edited by F. H. Busse and L. Kramer, NATO Advanced Study Institute, Ser. B2, Vol. 225 (Plenum, New York, 1990), p. 73.

[9] M. C. Cross, Phys. Rev. A **38**, 3593 (1988).

[10] D. P. Vallette, G. Jacobs, and J. P. Gollub, Phys. Rev. E **55**, 4274 (1997).

[11] J. F. Ravoux and P. Le Gal, Phys. Rev. E **58**, R5233 (1998).

[12] H. D. I. Abarbanel, R. Brown, J. J. Sidorowich, and L. S. Tsimring, Rev. Mod. Phys. **65**, 1331 (1993); H. Kantz and T. Schreiber, *Nonlinear Times Series Analysis* (Cambridge University Press, Cambridge, England, 1997).

[13] L. Breiman and J. H. Friedman, J. Am. Stat. Assoc. **80**, 580 (1985). We use a modified algorithm, where the data are rank ordered before the optimal transformations are estimated. This makes the result independent of the data distribution. Our implementation of the ACE algorithm is available on request from one of the authors (H. U. V.).

[14] A. Rényi, *Probability Theory* (Akadémiai Kiadó, Budapest, 1970).